Technical Sciences
Academy of Romania
www.jesi.astr.ro

Journal of Engineering Sciences and Innovation
Volume 11, Issue 1 / 2026, pp. 11 - 30

A. Mechanical Engineering

Received **27 August 2025** Accepted **11 March 2026**
Received in revised form **08 December 2025**

A new approach to the modal analysis of beams made of functionally graded materials

NASTASESCU VASILE[1*], BADEA MIHAELA[2], HADAR ANTON[2]

[1]Military Technical Academy „Ferdinand I", Bucharest, Romania
[2]National University of Science and Technology Politehnica Bucharest, Romania

Abstract. The paper presents the results of my own research on the calculus of structural elements – in this case beams – made of functionally graded materials, using an original approach, based on the concepts of equivalent material and multilayer material. The research carried out is illustrated by using of the power law of the functionally graded material, but the results can be generalized for any other material law. The numerical calculus is performed using the finite element method, and the results are presented in comparison with the analytical solution. The novelty brought by this paper results from the novelty of the structure material, which requires an appropriate calculation that takes into account all the characteristics of functionally graded materials. The novelty also lies in the original way of approaching the calculation, thus becoming accessible with current computing means.

Keywords: functionally graded material, equivalent material, multilayer material.

1. Introduction

The interest in the use of functionally graded materials is constantly growing, as a result of their behavior in various situations. The use of functionally graded materials is based on research carried out in the field of production of functionally graded materials and equally on research on the calculation of structures from functionally graded materials.
This paper is the result of my own scientific research in the field of calculation of structures from functionally graded materials. The structural element presented in this paper is a straight beam, of the Euler-Bernoulli type, loaded in bending.
The calculation to which the research presented in this paper refers is modal analysis, that is, the determination of the natural frequencies and the forms of the natural vibrations. The calculation proposed for the analysis of free vibrations of

*Correspondence address: nastasescuv@gmail.com

beams made of functionally graded materials is an original one, based on the concepts of equivalent material and multilayer material.
The beam is perhaps the most widely used structural element, both in mechanical and civil and industrial engineering, and modal analysis is of particular importance in the study of the behavior of structures in dynamic regimes.
Other aspects of the calculation of structures made of functionally graded materials – the fruit of my own research – have been published in a series of articles in specialized journals, but they also constituted the essence of a book published in Romanian [1]. The novelty brought by this article comes from the novelty regarding the material and the originality of the proposed method, models and calculation methodologies, which offer the accessibility, efficiency and precision specific to design or verification calculations in mechanical engineering.

2. Functionally Graded Materials

Functionally graded materials are those materials whose elastic, physical and mechanical properties are no longer expressed by material constants but by their variation functions in a certain direction [2], [3]. Therefore, parameters such as Young's modulus $E = E(y)$, density $\rho = \rho(y)$, Poisson's ratio $\nu = \nu(y)$, etc.

The limits of variation of the properties are represented by the values of two materials, which are the basis for the realization of the functionally graded material [4], [5], [6]. The two materials have very different properties, which are found on the extreme faces of the material, as in Figure 1.

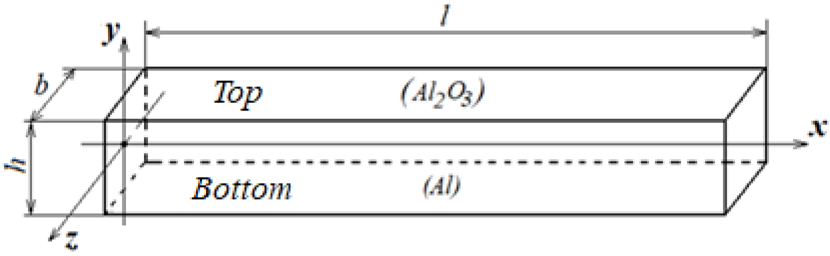

Fig. 1. Components of functional graded material.

The case presented in Figure 1 is that of a functionally graded material, made on the basis of a ceramic material (alumina) and a metal (aluminum), in the form of a beam having length $l = 0.20$ m, with section dimensions $b = h = 0.002$ m. The laws of variation of material properties are called material laws or models and can be different, according to the destination and requirements of such functionally graded materials. The most widely used material law – also used in our research – is the power law, expressed by relation (1). The most important parameter of the law is the power coefficient k.

$$E(z) = E_b + (E_t - E_b)\left(0.5 + \frac{y}{h}\right)^k \tag{1}$$

The indices b and t denote the bottom ($b \equiv bottom$) and top ($t = top$) surfaces, respectively. Relation (1) refers to Young's modulus, but the same law also defines the variation of density - relation (2) or of Poisson's ratio - relation (3).

$$\rho(z) = \rho_b + (\rho_t - \rho_b)\left(0.5 + \frac{y}{h}\right)^k \tag{2}$$

$$\nu(z) = \nu_b + (\nu_t - \nu_b)\left(0.5 + \frac{y}{h}\right)^k \tag{3}$$

Qualitatively, the power function has variation in Figure 2, for the three ranges of variation of the power coefficient k, which even determines a change in curvature.

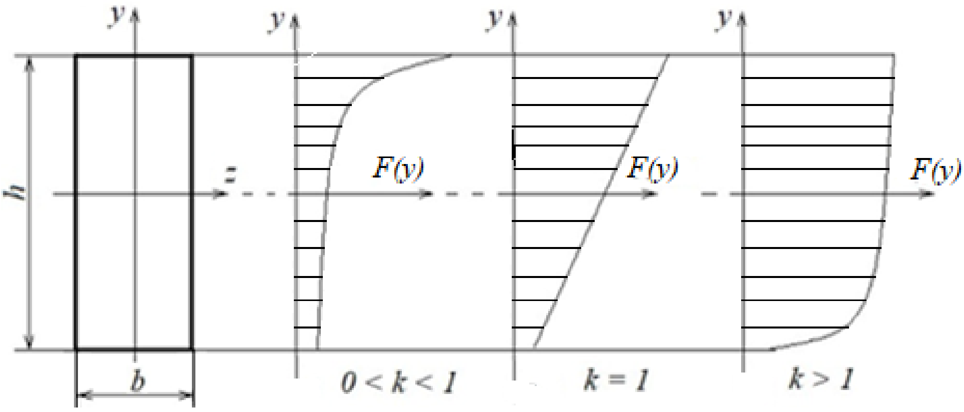

Fig. 2. The shape of the power law curve as a function of the coefficient *k*.

Our research took into account the properties of the two materials, whose properties are presented in Table 1.

Table 1. Material properties

Materials	Ceramic material (Al_2O_3)	Aluminum (Al)
Position	Top	Bottom
E [Pa]	$3.8 \cdot 10^{11}$	$7 \cdot 10^{10}$
ν [−]	0.22	0.33
ρ [kg/m^3]	3960	2700

Using relations (1)...(3) the curves of variation of material properties over the section height (h , Figure 1) as a function of the power coefficient k are those presented in Figures 3, 4 and 5.

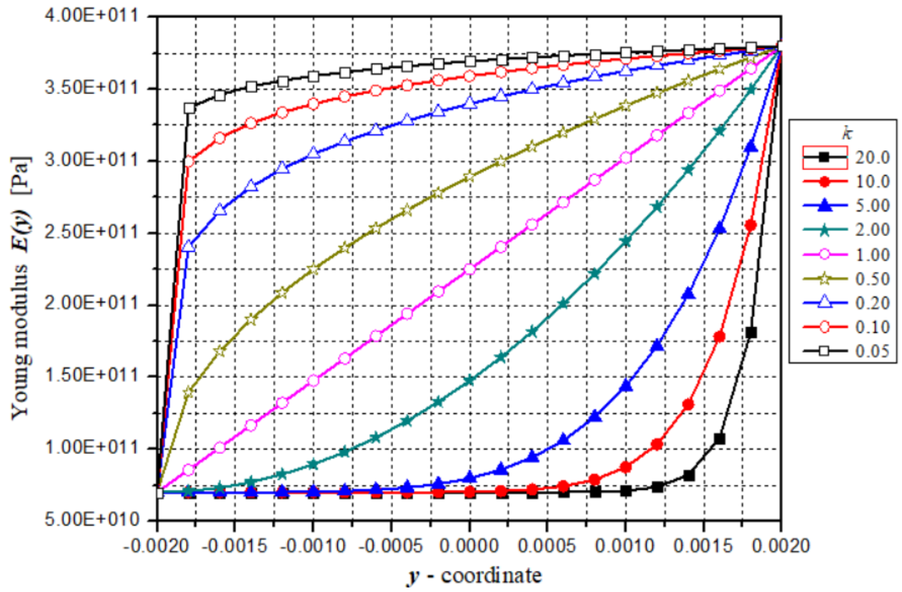

Fig. 3. Variation of Young's modulus over the height of the section.

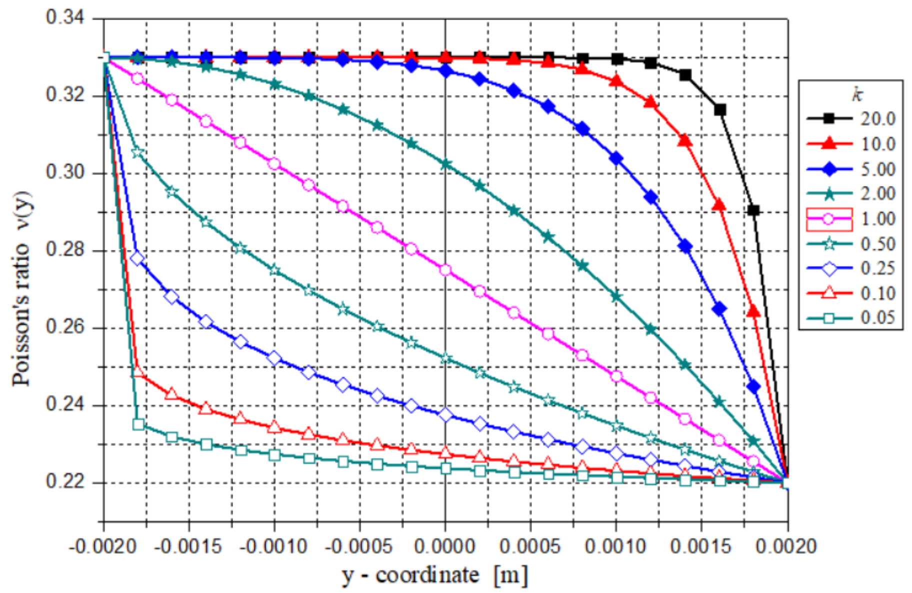

Fig. 4. Variation of Poisson's ratio over the height of the section.

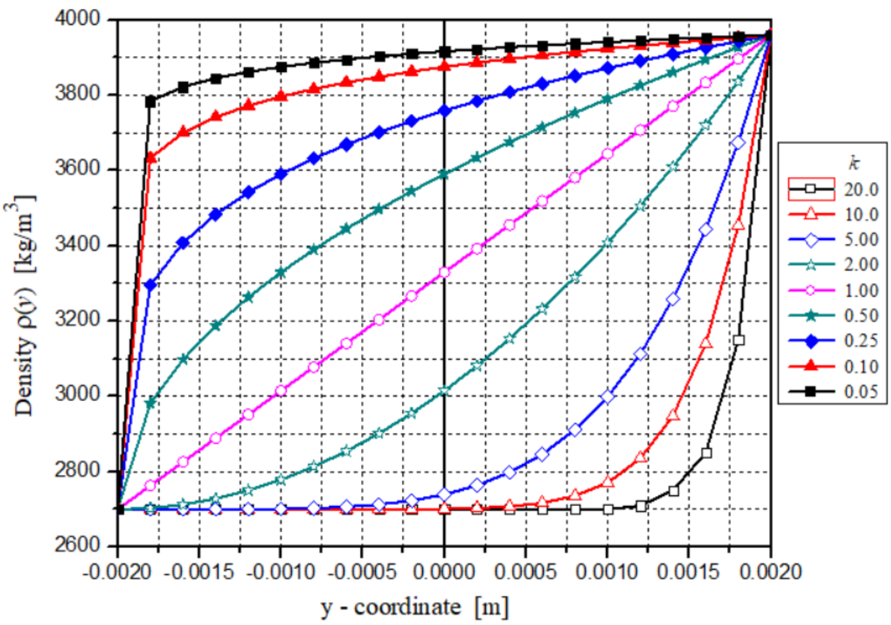

Fig. 5. Density variation over the height of the section.

The analysis of the curves in Figures 3, 4 and 5 shows us that the shape of the curves $E(y)$ and $\rho(y)$ is the same, but with different values, and the shape of the curve $\nu(y)$ is different due to the extreme values of the two materials.

It is also found that small values of the power coefficient ($k<1$) lead to an increased influence of the material with superior characteristics (the ceramic material in this case). For the value of 1 of the power coefficient, the variation of all properties is linear. Figures 3, 4 and 5 also show us a different curvature of the material properties variation curves, for the power coefficient value ranges: $k<1$, respectively $k>1$.

3. Original Concepts of Approaching Calculus

For the calculation of structures made of functionally graded materials, two concepts are used: the concept of multilayer material [7] and the concept of equivalent material [8], which can be applied separately or in combinations, both in analytical calculation and in numerical calculation with the finite element method and even with meshless (SPH – Smoothed Particle Hydrodynamics) or meshfree (Galerkin free element) methods.

3.1. The Concept of Multilayer Material

The concept of multilayer material considers that the functionally graded material is composed of a number of layers of different materials, with homogeneous and isotropic properties (Figure 6) [7], [9], [10]. Each layer (j) is assigned the material properties calculated with the relations (1)...(3). For an infinite number of layers, the variation of the properties becomes continuous, according to the adopted law.

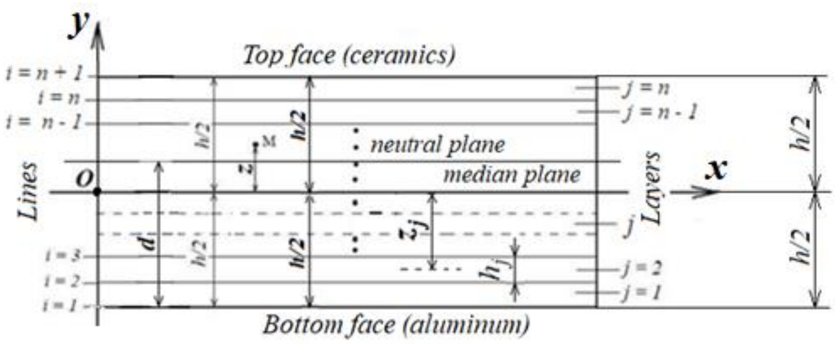

Fig. 6. Illustration of the multilayer material concept.

In practice, one must work with a finite number of layers. This concept causes the continuous variation of the properties to be replaced by a step variation, as exemplified in Figure 7, regarding the variation of the Young's modulus.

.

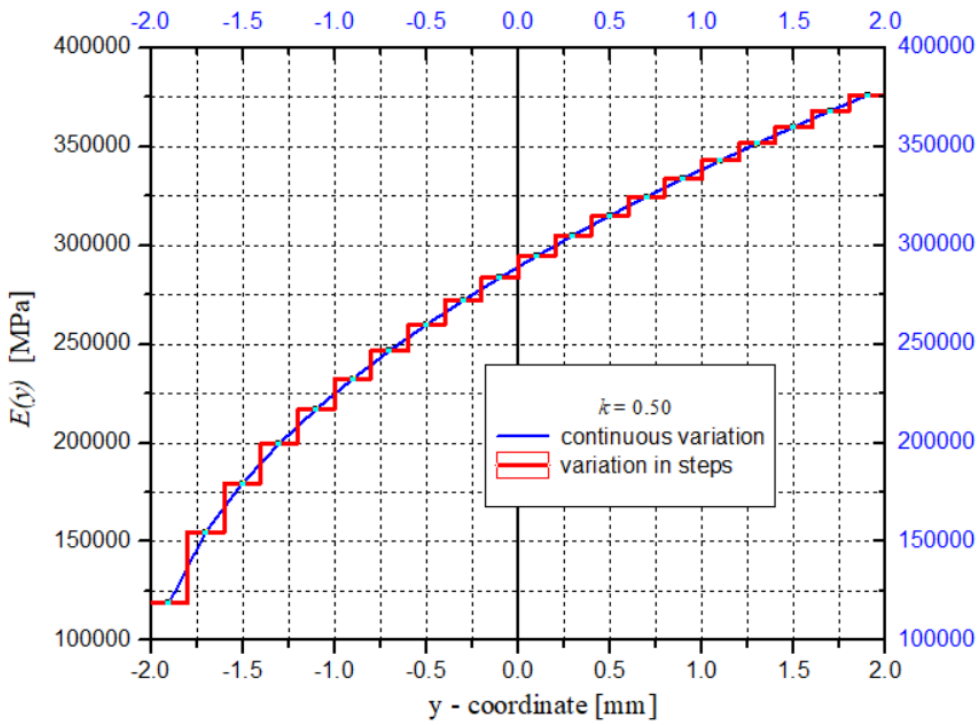

Fig. 7. Stepwise variation of Young's modulus.

Our research on this aspect has allowed us to quantitatively assess the influence of the number of layers on the results of the strength calculation or modal analysis [9], [10], [11], [12]. Both our own research and the experience gained have allowed us to establish an empirical calculation relationship (4) for calculating the maximum thickness $t_{\max}$ of the layers.

$$t_{\max} \leq \frac{h}{20}[m] \tag{4}$$

where h is the section height expressed in meters. Knowing the thickness of the layers, their number immediately results. Applying relation (4) for the case study in this paper, leads to a minimum number of 20 layers, each with a thickness of 0.0002 m. The graphical representations in Figures 8, 9 and 10 illustrate this concept.

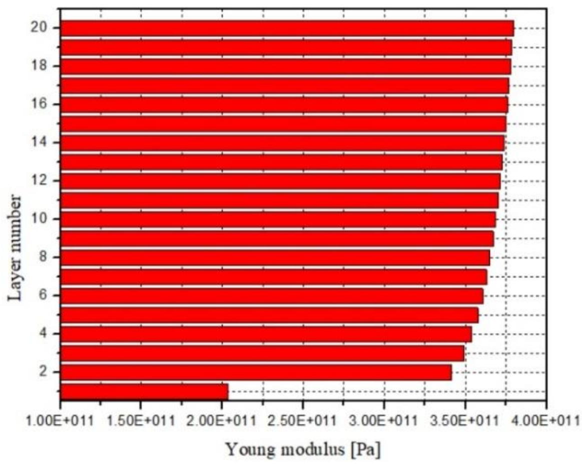

Fig. 8. Variation of Young's modulus on material layers, for $k = 0.05$.

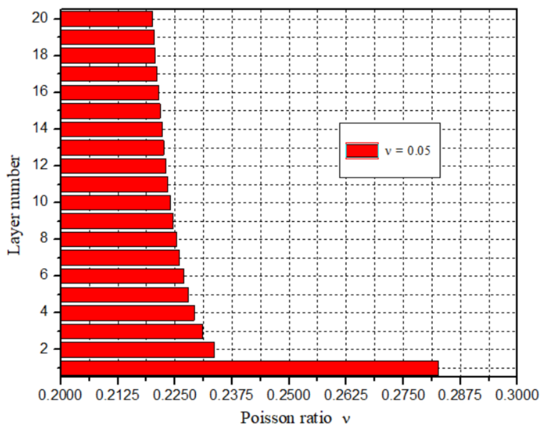

Fig. 9. Variation of Poisson's ratio on material layers, for $k = 0.05$.

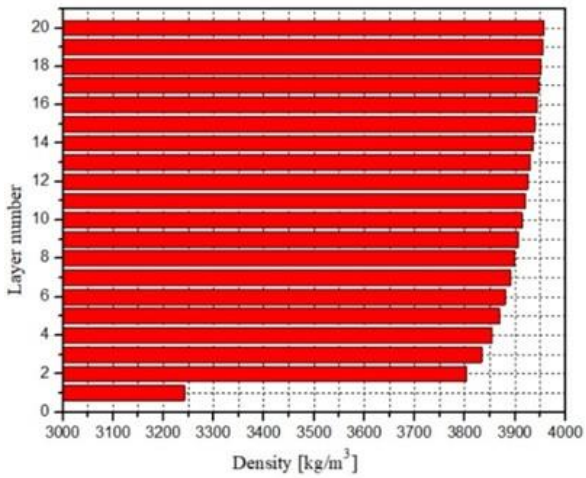

Fig. 10. Density variation on material layers, for $k = 0.05$.

Table 2. The values of elastic constants at the level of lines and layers, for $k = 0.05$

Lines	*E(y)*	*ρ(y)*	*ν(y)*	Layers	*E(y)*	*ρ(y)*	*ν(y)*
1	7.000E+10	2700.00	0.330	1	3.278E+11	3747.77	0.311
2	3.369E+11	3784.72	0.235	2	3.423E+11	3806.94	0.317
3	3.463E+11	3822.98	0.232	3	3.494E+11	3835.58	0.319
4	3.519E+11	3845.97	0.230	4	3.541E+11	3854.84	0.321
5	3.560E+11	3862.58	0.229	5	3.577E+11	3869.44	0.322
6	3.592E+11	3875.62	0.227	6	3.606E+11	3881.24	0.323
7	3.619E+11	3886.39	0.226	7	3.631E+11	3891.15	0.324
8	3.641E+11	3895.57	0.226	8	3.652E+11	3899.70	0.325
9	3.661E+11	3903.58	0.225	9	3.670E+11	3907.23	0.325
10	3.679E+11	3910.69	0.224	10	3.687E+11	3913.96	0.326
11	3.694E+11	3917.08	0.224	11	3.702E+11	3920.05	0.327
12	3.709E+11	3922.89	0.223	12	3.715E+11	3925.61	0.327
13	3.722E+11	3928.23	0.223	13	3.728E+11	3930.73	0.327
14	3.734E+11	3933.15	0.222	14	3.740E+11	3935.48	0.328
15	3.745E+11	3937.73	0.222	15	3.751E+11	3939.90	0.328
16	3.756E+11	3942.01	0.222	16	3.761E+11	3944.04	0.329
17	3.766E+11	3946.02	0.221	17	3.770E+11	3947.94	0.329
18	3.775E+11	3949.80	0.221	18	3.779E+11	3951.62	0.329
19	3.784E+11	3953.38	0.221	19	3.788E+11	3955.10	0.330
20	3.792E+11	3956.77	0.220	20	3.796E+11	3958.41	0.330
21	3.800E+11	3960.00	0.220	-	-	-	-
Average values	**3.669E+11**	**3849.29**	**0.230**	**Average values**	**3.654E+11**	**3900.84**	**0.325**

The values on which the graphical representations in Figures 8, 9 and 10 were made are presented in Table 2, based on relations (1)...(3), for the y coordinate corresponding to the lines and the average values of the lines defining each layer (Figure 6).

3.2. The Concept of Equivalent Material

The concept of equivalent material consists in replacing the functionally graded material with a homogeneous and isotropic one, with fictitious properties, established from the requirement of an equivalent behavior to the given material [8]. For the calculation of natural frequencies [13], [14], [15], displacements and stresses [16], [17], [18], the fictitious (equivalent) properties are Young's modulus, density and Poisson's ratio.

3.2.1. The Calculus of Equivalent Density

The infinitesimal mass element of the bar is written:

$$\mathrm{dM} = \rho \cdot \mathrm{dV} = \rho(y) \cdot \mathrm{A} \cdot \mathrm{d\,y} = \rho(y) \cdot bl \cdot \mathrm{d\,y} \tag{5}$$

In relation (5) the density is introduced with the expression of the used material law. In our research the power law was used, which leads to the following results.

$$\mathrm{M} = \int \mathrm{dM} = bl \cdot \int_{-h/2}^{h/2} \rho(\mathrm{y})\,\mathrm{d\,y} \tag{6}$$

By introducing relation (2) into (6) and performing the calculations, we obtain:

$$\begin{aligned} \mathrm{M} &= b \cdot l \int_{-h/2}^{h/2} \left[\rho_b + (\rho_t - \rho_b) \cdot \left(0{,}5 + \frac{y}{h} \right)^k \right] \mathrm{d\,y} = \\ &= b \cdot l \cdot \left[\rho_b \cdot h + (\rho_t - \rho_b) \frac{h}{k+1} \right] = b \cdot l \cdot h \left[\rho_b + (\rho_t - \rho_b) \frac{1}{k+1} \right] \end{aligned} \tag{7}$$

Because,

$$\mathrm{M} = \mathrm{V} \cdot \rho_{ech} = b \cdot l \cdot h \tag{8}$$

the calculus relation of equivalent density (ρ_{ech}) results:

$$\rho_{ech} = \rho_b + \frac{\rho_t - \rho_b}{k+1} \tag{9}$$

Using relation (9), for a functionally graded material made based on the data in Table 1, we obtain the equivalent density values presented in Table 3, which form the basis of the graphical representation in Figure 11.

Table 3. Comparative values of equivalent density

Functionally graded material			
k	ρ_b	ρ_t	ρ_{ech}
-	kg/m^3	kg/m^3	kg/m^3
0.05	2700	3960	3900.0
0.10	2700	3960	3845.5
0.25	2700	3960	3708.0
0.50	2700	3960	3540.0
1.00	2700	3960	3330.0
2.00	2700	3960	3120.0
4.00	2700	3960	2952.0
6.00	2700	3960	2880.0
8.00	2700	3960	2840.0
10.00	2700	3960	2814.5
15.00	2700	3960	2778.8
20.00	2700	3960	2760.0

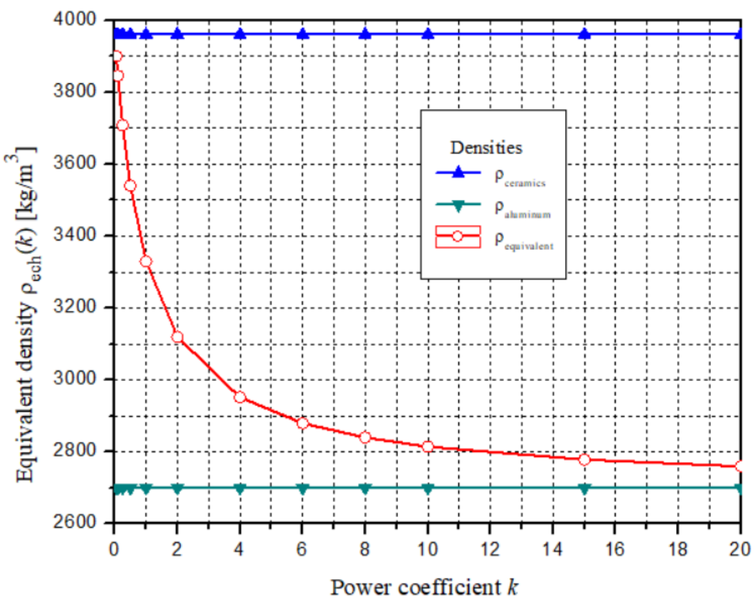

Fig. 11. Variation of equivalent density with power coefficient *k*.

From the analysis of the results and the graphical representation, it is found that the equivalent density values decrease with increasing power coefficient k, and the equivalent density values, for any values of the power coefficient, are between the values of the base materials (ρ_b, ρ_t).

3.2.1. Calculus of the equivalent Young's modulus

Considering the well known relationships (10) and (11) for a bent right bar,

$$\frac{1}{\rho} = \frac{\mathrm{d}\varphi}{\mathrm{dx}} = \omega = \frac{M}{EI} \tag{10}$$

$$\sigma = E \cdot \omega \cdot y \tag{11}$$

the equivalence relation of the bending moment in the section is written:

$$M = \int_{-h/2}^{h/2} \sigma \cdot \mathrm{dA} \cdot y = \int_{-h/2}^{h/2} E(y) \cdot \omega \cdot y^2 \cdot b \cdot \mathrm{d\,y} \tag{12}$$

Since rotation has a constant value per section, relation (12) becomes:

$$M = \omega b \cdot \int_{-h/2}^{h/2} E(y) \cdot y^2 \cdot \mathrm{d\,y} = \omega b \cdot \mathrm{I}^* \tag{13}$$

Solving the integral,

$$\mathrm{I}^* = \int_{-h/2}^{h/2} E(y) \cdot y^2 \cdot \mathrm{d\,y} \tag{14}$$

leads to the following result:

$$\mathrm{I}^* = E_b \frac{h^3}{12} + (E_t - E_b) h^3 \left[\frac{1}{3+k} - \frac{1}{2+k} + \frac{1}{4(k+1)} \right] \tag{15}$$

With this result, relation (13) becomes:

$$M = \omega b \cdot \left\{ E_b \frac{h^3}{12} + (E_t - E_b) h^3 \left[\frac{1}{3+k} - \frac{1}{2+k} + \frac{1}{4(k+1)} \right] \right\} \tag{16}$$

From relation (16), it follows:

$$\omega = \frac{M}{\left\{ E_b + 12(E_t - E_b) \left[\frac{1}{3+k} - \frac{1}{2+k} + \frac{1}{4(k+1)} \right] \right\} \cdot I} \tag{17}$$

By identifying in relations (10) and (17), the equivalent Young's modulus is:

$$E_{ech} = E_b + 12(E_t - E_b)\left[\frac{1}{3+k} - \frac{1}{2+k} + \frac{1}{4(k+1)}\right] \tag{18}$$

Table 4. Comparative values of Young's modulus

Functionally graded material			
k	E_b	E_t	E_{ech}
0.05	7.00E+10	3.80E+11	3.608E+11
0.10	7.00E+10	3.80E+11	3.440E+11
0.25	7.00E+10	3.80E+11	3.053E+11
0.50	7.00E+10	3.80E+11	2.649E+11
1.00	7.00E+10	3.80E+11	2.250E+11
2.00	7.00E+10	3.80E+11	1.940E+11
4.00	7.00E+10	3.80E+11	1.674E+11
6.00	7.00E+10	3.80E+11	1.512E+11
8.00	7.00E+10	3.80E+11	1.395E+11
10.00	7.00E+10	3.80E+11	1.307E+11
15.00	7.00E+10	3.80E+11	1.160E+11
20.00	7.00E+10	3.80E+11	1.069E+11

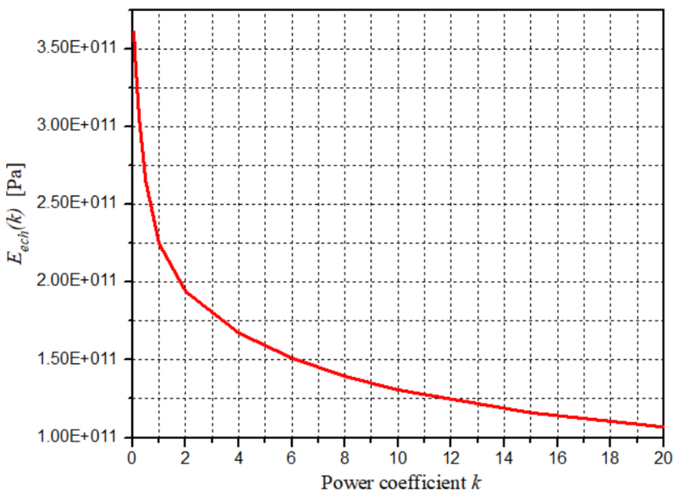

Fig. 12. Variation of equivalent Young's modulus to the power coefficient *k*.

For a functionally graded material made based on the data in Table 1, the equivalent Young's modulus values presented in Table 4 result, which form the basis of the graphical representation in Figure 12.

It is found that with the increase in the power coefficient *k*, the values of the equivalent Young's modulus decrease, and its values, for any value of *k*, are found between the values E_b and E_t.

Regarding the Poisson's ratio, the authors of many articles on the calculation of structures made of functionally graded materials state that its variation can be neglected. In our research, the influence of the variation of the Poisson's ratio was also analyzed [18].
The use of the concepts proposed by us can also lead to the calculation of an equivalent Poisson's ratio, as the average of its values for the number of layers considered, as will be presented in the example for the calculation of the natural frequencies of straight beams made of functionally graded materials.

4. Analytical calculus of eigenfrequencies

The analytical calculation of the eigenfrequencies for a homogeneous and isotropic straight bar is a classical problem in vibration theory [13], [14], [15], which can lead to the determination of both the eigenfrequencies and their shapes.
Starting from the equilibrium of the infinitesimal element of the bar (Figure 9), considering the differential relations between the stresses and the differential equation of the deformed mean fiber, we arrive at the equation,

$$EI\frac{\mathrm{d}^4 \mathrm{v}}{\mathrm{dx}^4} = p \tag{19}$$

where the distributed load is the inertia force that determines the lateral displacements of the bar. This has the expression:

$$p = -\rho A\frac{\mathrm{d}^2 \mathrm{v}}{\mathrm{d}t^2} \tag{20}$$

Introducing relation (20) into (19) and using the notation,

$$\alpha^2 = \frac{EI}{\rho A} \tag{21}$$

the following differential equation is reached:

$$\frac{\mathrm{d}^2 \mathrm{v}}{\mathrm{d\,t}^2} + \alpha^2 \cdot \frac{\mathrm{d}^4 \mathrm{v}}{\mathrm{dx}^4} = 0 \tag{22}$$

In the case of normal (natural) vibrations, the displacement of any section varies harmonically in time, and the solution of equation (22) can have the form [13], [16],

$$\mathrm{v}(x,t) = F(x)\cdot\left(A\cos pt + B\sin pt\right) \tag{23}$$

where $F(x)$ is a function of x, which describes the shape of the vibration mode. Through a series of mathematical operations, the relationship is reached:

$$\frac{\mathrm{d}^4 F}{\mathrm{dx}^4} = \frac{p^2}{\alpha^2} F \tag{24}$$

The differential equation (24) has the solution,

$$F(x) = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx \tag{25}$$

where C_1, C_2, C_3 and C_4 are integration constants, which can be determined from the boundary conditions (the beam's support mode).

Once the integration constants are determined, both the natural frequencies and the vibration shapes can be determined analytically. If we consider all the eigenmodes of free (natural) vibrations, the general solution of free transverse vibrations is written [13], [16]:

$$\mathrm{v}(x,t) = \sum_{i=1}^{i=\infty} F_i(x) \cdot \left(A_i \cos p_i t + B_i \sin p_i t \right) ... \tag{26}$$

Of course, in practice one works with a finite number of eigenmodes of vibration, so the parameter "i" will be a finite parameter "n".

For the simply supported beam, after determining the integration constants, the following calculus relationship for the eigenfrequencies is reached,

$$f_n = \frac{\pi n^2}{2l^2} \cdot i \cdot \sqrt{\frac{E}{\rho}} \tag{27}$$

where is the natural frequency number ($n = 1, 2, 3, 4, 5, ...$) and i is the radius of gyration of the section. For the beam made of a functionally graded material, by applying the concept of equivalent material, relation (27) becomes:

$$f_n = \frac{\pi n^2}{2l^2} \cdot i \cdot \sqrt{\frac{E_{ech}}{\rho_{ech}}} \tag{28}$$

Using relation (28) leads to the results in Tables 5 and 6 and also the graphical representations in Figure 13, regarding the natural frequencies of a simply supported beam, made of functionally graded material based on the power material law, for several values of the power coefficient *k*.

Table 5. The values of the eigenfrequencies depending on the coefficient *k*.

k	f_1	f_2	f_3
	Hz	Hz	Hz
0.05	436.12	1744.46	3925.04
0.10	428.89	1715.58	3860.05
0.25	411.44	1645.77	3702.99
0.50	392.22	1568.89	3530.01
1.00	372.73	1490.93	3354.60

k	f_1	f_2	f_3
	Hz	Hz	Hz
2.00	357.56	1430.25	3218.07
4.00	341.50	1365.98	3073.46
6.00	328.54	1314.18	2956.90
8.00	317.82	1271.28	2860.37
10.00	309.00	1236.01	2781.02
15.00	292.94	1171.75	2636.43
20.00	282.25	1129.00	2540.24

Table 6. Natural frequencies of the homogeneous and isotropic bar.

Aluminum (AL)		
f_1	f_2	f_3
Hz	Hz	Hz
230.89	923.54	2077.97
Ceramics (Al_2O_3)		
f_1	f_2	f_3
Hz	Hz	Hz
444.19	1776.78	3997.75

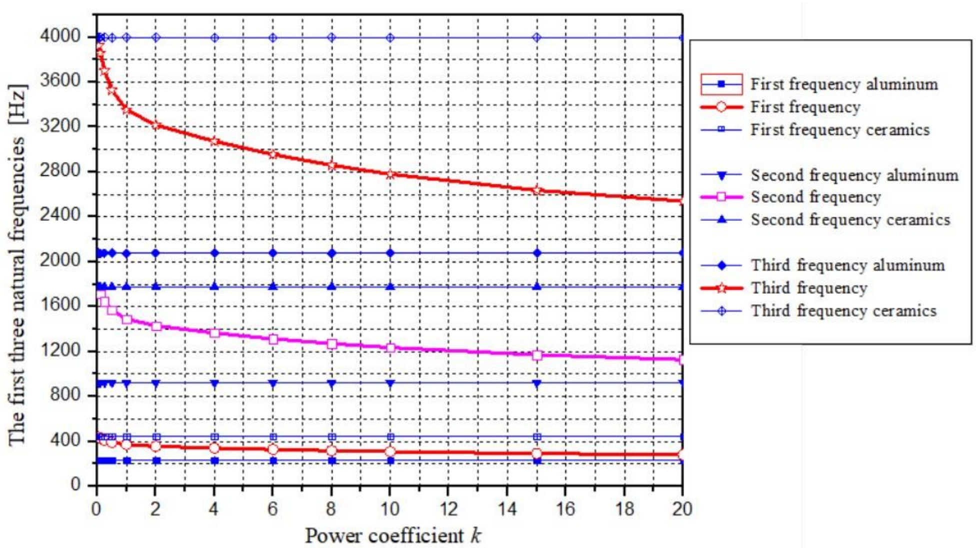

Fig. 13. Variation of natural frequencies with power coefficient *k*.

The first three natural frequencies of the bar considered, if it were made of homogeneous and zoetropic base materials (aluminum and ceramics), are presented in Table 6, calculated based on relation (27).

For an easier comparative analysis, based on the data in Tables 5 and 6, the frequency variation curves are presented in Figure 13. Analysis of the graphic representation in Figure 13 shows us that the difference between the extreme values of the eigenfrequencies increases with the number of the eigenfrequency. It is also found that for any value of the power coefficient k, the values of the natural frequencies of the bar made of functionally graded material are found between the values of the natural frequencies of the bar made of basic materials.

5. Numerical calculation of natural frequencies

The numerical calculation is performed with the finite element method using the Ansys program and the general theory regarding the calculation by the finite element method [19], [20], [21].

5.1. The model based on the concept of equivalent material

This finite element model is based on the use of the 8-node brick finite element, called Solid185, from the finite element library of the Ansys program, [22], [23] with the options: solid structural element, displacement formulation and full integration.

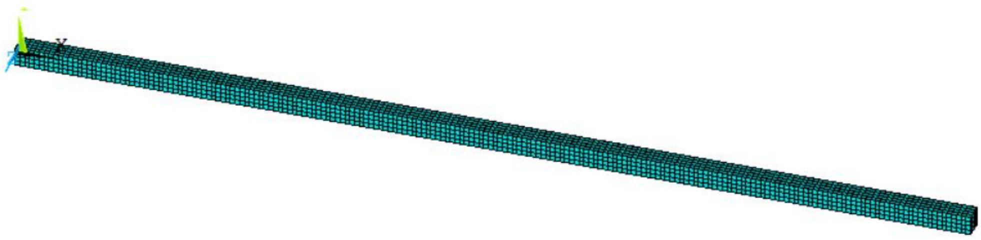

Fig. 14. Finite element model of the functionally graded beam.

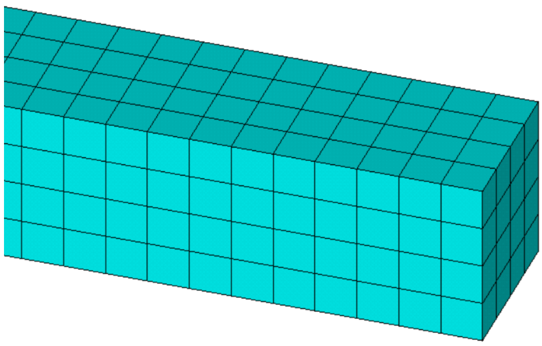

Fig. 15. Detail of the finite element model.

As can be seen in figures 14 and 15, the finite element is represented by a cube, with a side of 1 mm. The model contains 5025 nodes and 3200 finite elements.

The material model is the homogeneous and isotropic one with the fictitious properties of the equivalent material (Tables 2, 3 and 4).

5.2. Numerical calculation results

The results of the numerical calculation are presented in comparison with those of the analytical calculation, in tabular form (Tables 7...9), for several values of the power coefficient *k.* The use of the Ansys program also allowed for a graphical post-processing of the results, thus highlighting the vibration modes. The results and their comparison refer to the first three natural frequencies – the most frequently invoked in mechanical engineering calculations (the first natural frequency, also called the fundamental frequency, being the most important).

Table 7. The values of the first frequency and the factors that influence it

k	$f_1^{analytic}$	$\nu = \nu(\kappa)$		$\nu = 0.30$		$\nu = 0$	
		f_1^{FEM}	*Er.*	f_1^{FEM}	*Er.*	f_1^{FEM}	*Er.*
-	Hz	Hz	[%]	Hz	[%]	Hz	[%]
0.05	436.12	436.72	*0.14*	434.774	*-0.31*	424.365	*-2.69*
0.10	428.89	431.32	*0.57*	429.774	*0.21*	419.485	*-2.19*
0.25	411.44	416.48	*1.22*	415.983	*1.10*	406.024	*-1.32*
0.50	392.22	394.52	*0.59*	397.204	*1.27*	387.694	*-1.15*
1.00	372.73	367.74	*-1.34*	369.233	*-0.94*	360.394	*-3.31*

Table 8. The values of the second frequency and the factors that influence it

k	$f_2^{analytic}$	$\nu = \nu(\kappa)$		$\nu = 0.30$		$\nu = 0$	
		f_2^{FEM}	*Er.*	f_2^{FEM}	*Er.*	f_2^{FEM}	*Er.*
-	Hz	Hz	[%]	Hz	[%]	Hz	[%]
0.05	1744.46	1737.00	*-0.43*	1729.35	*-0.87*	1688.75	*-3.19*
0.10	1715.58	1715.50	*0.00*	1709.47	*-0.36*	1669.33	*-2.70*
0.25	1645.77	1657.50	*0.71*	1654.61	*0.54*	1615.77	*-1.82*
0.50	1568.89	1569.40	*0.03*	1579.91	*0.70*	1542.82	*-1.66*
1.00	1490.93	1462.80	*-1.89*	1468.66	*-1.49*	1434.18	*-3.81*

Table 9. The values of the third frequency and the factors that influence it

k	$f_3^{analytic}$	$\nu = \nu(\kappa)$		$\nu = 0.30$		$\nu = 0$	
		f_3^{FEM}	*Er.*	f_3^{FEM}	*Er.*	f_3^{FEM}	*Er.*
-	Hz	Hz	[%]	Hz	[%]	Hz	[%]
0.05	3925.04	3863.64	*-1.56*	3847.14	*-1.98*	3760.80	*-4.18*
0.10	3860.05	3816.00	*-1.14*	3802.90	*-1.48*	3717.55	*-3.69*
0.25	3702.99	3687.20	*-0.43*	3680.87	*-0.60*	3598.26	*-2.83*
0.50	3530.01	3492.00	*-1.08*	3514.70	*-0.43*	3435.82	*-2.67*
1.00	3354.60	3254.50	*-2.98*	3267.20	*-2.61*	3193.87	*-4.79*

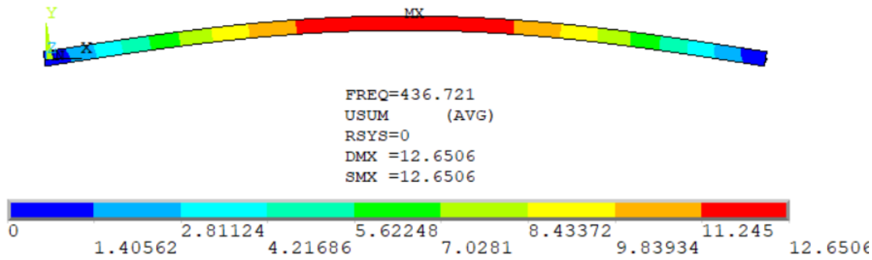

Fig. 16. The shape of the first eigenfrequency.

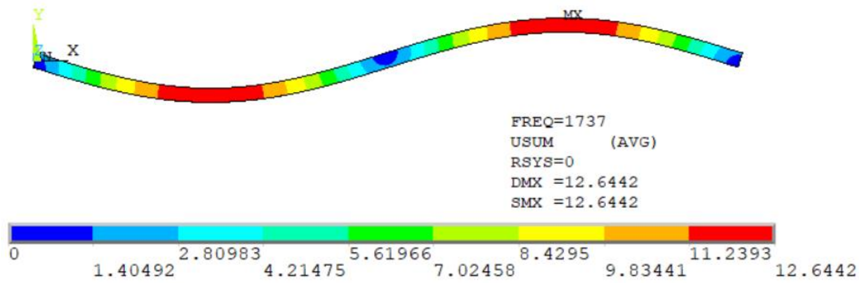

Fig. 17. The shape of the second natural frequency.

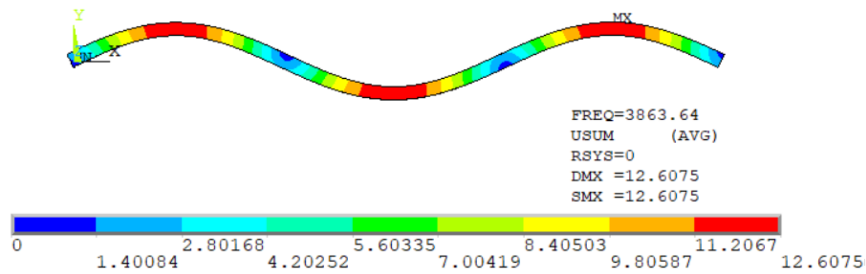

Fig. 18. The shape of the third natural frequency.

The use of numerical calculation also allowed taking into account the variation of the Poisson's ratio (Table 2) and establishing its influence. In the specialized literature, the influence of the variation of the Poisson's ratio is neglected (a constant value is used) for reasons of simplification of the calculation.
The analysis of the values in Tables 7...9 shows that the results of the numerical calculation are in very good agreement with the results of the analytical calculation.
With the increase in the power coefficient value, a slight increase in errors is also observed. This aspect is explained by the variation mode of the parameters E, ρ, ν (Figures 3, 4 and 5) whose variation is accentuated with the increase in the power coefficient; then, the adopted finite element models are based on a uniform thickness of the layers. Since the errors are acceptable (in their vast majority around the value of 1% and less than 1%), the finite element model can be considered acceptable, representing a valid model.

6. Conclusions

The paper deals, in an original manner, with a topic of great relevance and importance for mechanical engineering – the calculation of beams made of functionally graded materials.
The originality and consequently the novelty brought by the work lies in the manner of solving, by using the concepts of equivalent material and multilayer material. These concepts can be used both independently and in combinations, leading to the greatest possible efficiency of the calculation.
What these concepts consist of and how they are practically used in analytical or numerical engineering calculation are aspects treated in detail and by example in the paper. The use of numerical calculation based on the two concepts allowed a comparative analysis of the results when the Poisson's ratio has the value zero, has a constant value and when it varies according to the material law of the functionally graded material.
The use of numerical calculation, based on the two concepts, allows the variation of Poisson's coefficient values to be taken into account.
The numerical approach based on the two concepts allows its use using common finite element analysis programs without any modification in the definition of the finite element matrices. Thus, the concepts, models and methodology presented are immediately accessible to any researcher or designer of structures made of functionally graded materials.
The comparative presentation of analytical and numerical results, with the presentation of the errors of the numerical results compared to the analytical ones, highlights the efficiency and accuracy of the modal analysis of beams made of functionally graded materials.

References

[1] Nastasescu V., Barsan Gh., Mocian O., *Calculul unor elemente structurale din materiale gradate funcţional*, Editura Academiei Române, Bucureşti, 2025, ISBN: 978-973-27-3950-1.
[2] Anju M., Subha K., *A Review on Functionally Graded Plates*, Int. Res. J. Eng. Technol., 2018, 5.
[3] Toudehdehghan A., Lim J.W., Foo K.E., Ma'Arof M., Mathews J., *A Brief Review of Functionally Graded Materials*, MATEC Web Conf., 2017, 131, 03010. [CrossRef]
[4] Pradhan K.K., Chakraverty S., *Overview of functionally graded materials*, Computational Structural Mechanics, Elsevier, Amsterdam, The Netherlands, 2019, p. 1–10.
[5] Mahamood R.M., Akinlabi E.T., Shukla M., Pityana S.L., *Functionally graded material: An overview*, Proceedings of the World Congress on Engineering, London, UK, 4–6 July 2012, Volume 3.
[6] Nastasescu V., Marzavan S., *An overview of functionally graded material models*, Proceedings of the Romanian Academy— Series A, Bucharest, Romania, 20 September 2022, Volume 23, p. 257–265.
[7] Năstăsescu V., *Research and Developments in Engineering Research*, Volume 8, Chapter 1: Study about Calculus of Functionally Graded Plates by using Multylayer Plate Concept, Print ISBN 978 81-19761-42-5, eBook ISBN 978-81-19761-53-1, DOI: 10.9734/bpi/rader/v8, Print ISBN: 978-81 19217-45-8, eBook ISBN: 978-81-19217-47-2, DOI: 0.9734/bpi/rader/v3/19159D.
[8] Năstăsescu V., Marzavan S., *Using the Equivalent Plate Concept to Calculate Functionally Graded Plates by Numerical Method*, IOP Conf. Series: Materials Science and Engineering, 1262, ACME_Iaşi, June 9–10 (2022), 012056, DOI: 10.1088/1757-899X/1262/1/012056.
[9] Szilard R., *Theories and Applications of Plate Analysis: Classical, Numerical and Engineering Methods*, Appl. Mech. Rev., **57**, 2004, B32–B33. [CrossRef].
[10] Timoshenko S. P., *Theory of Elasticity*, 3rd Edition, Tokyo, London, New York, McGraw-Hill, 1970.
[11] Bia C., Ille V., Soare M. V., *Rezistenţa materialelor şi teoria elasticităţii*, Bucureşti, Editura Didactică şi Pedagogică, 1983.
[12] Bassin M.G., Brodsky S.M., Wolkoff H., *Statics and Strength of Materials*, Third Edition, McGraw-Hill Book Company, 1979.
[13] Timoshenko S*., Vibration Problems in Engineering*, 2nd ed.; D. Van Nostrand Company, Inc.: New York, NY, USA, 1937.
[14] Leissa W.A., *Vibration of Plates*, National Aeronautics and Space Administration: Washington, DC, USA, 1969.
[15] Alshorbag A.E., Eltaher M., Mahmoud F., *Free Vibration Characteristics of a Functionally Graded Beam by Finite Element Method*, Appl. Math. Model, **35**, 2010, p. 412–425. [CrossRef].
[16] Timoshenko S. P., *Theorie des Vibrations* (traduit d'après la deuxième édition), Paris, 1954.
[17] Ramu I., Mohanty S., *Modal Analysis of Functionally Graded Material Plates Using Finite Element Method*, Procedia Mater. Sci., **6**, 2014, p. 460–467. [CrossRef].
[18] Năstăsescu V., *Research and Developments in Engineering Research*, Volume 3, Chapter 9: The Influence of Poisson's ratio in the Calculus of Functionally Graded Plates, Print ISBN: 978-81-19217-45-8, eBook ISBN: 978-81-19217-47-2, DOI: 0.9734/bpi/rader/v3/19159D.
[19] Bathe K. J., *Finite Element Procedures*, Prentice-Hall, Inc.: Hoboken, NJ, USA, 1996.
[20] Huebner K.H., Dewhirst D.L., Smith D.E., Byrom T.G., *The Finite Element Method for Engineers*, 3rd ed.; John Willey and Sons, Inc.: Hoboken, NJ, USA, 1995.
[21] Zienkiewicz O.C., Taylor R.L., Zhu J.Z., *The Finite Element Method*, 4th ed.; McGraw-Hill Book Company, New York, NY, USA, 2005.
[22] ***, *Theory Reference for the Mechanical APDL and Mechanical Applications*, Ansys Inc., Release 12, 2009
[23] ***, *Ansys Theory Reference*, Ansys Inc., Release 5.6, 1999