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Development of the precessional transmissions from invention to applications

ION BOSTAN, ALEXANDRU BUGA*

Technical University of Moldova, Chisinau, Republic of Moldova

Abstract. This article presents theoretical and applied contributions regarding the development of teethed precessional transmissions with convex-concave multipair contact $A^{D}{}_{CX\text{-}CV}$. Is described the mathematical model of the $A^{D}{}_{CX\text{-}CV}$ gearing and identifying the dependence of the contact geometry on the $[Z_g\text{-}\Theta, \pm 1]$ configuration parameters. Also, are presented the envelope equation of the arcs on the sphere with radius followed by the sequence of conjugation of flank profiles of the same pair of teeth in the k_i contacts with ($\mathbf{r}_{ki}$-r) characteristics depending on the $\mathbf{y}_{ki}$ precession angle in the $A^{D}{}_{CX\text{-}CV}$ and $A^{D,b}{}_{CX\text{-}CV}$ gears. Are presented the dispersion of contact points and the evolution of the pressure angle between the flanks.

Keywords: development of precessional transmissions, mathematical model, the equation of the envelope, precession angle.

1. Introduction

This paper is original by the innovative solutions proposed for the constructive-functional development of precessional planetary transmission *(PPT)*, by the complex approach of the problems and the concepts of development, as well as by the methods regarding their solution [1]. Also, include research in the thematic spectrum from structural concepts of *PPT* to theoretical and experimental approaches on physical models, from non-standardized forms of flank profiles of the teeth to the elaboration of their generation processes, from the constructive-functional design of different transmission to their industrial application, [2],[3]. The novelty of geometry development of the convex/concave flank profiles of the central wheel teeth, which ensures a concave/concave contact with a small difference of the radii of curvature and the reduction of the frictional sliding between the conjugated flanks, [4].

*Correspondence address: alexandru.buga@precesia.utm.md

2. Tooth profile expressed by the wrap of the circle arc family

For the tooth profile description of the central wheel is necessary to draw circular arcs with radius EG -red line, fig. 1.

The coordinates are transferred of the origin of the circular arc radius $\boldsymbol{G}$ into the mobile coordinate system $O\bar{X}\bar{Y}\bar{Z}$ linked to the central wheel according to the relations:

$$\begin{gathered}\bar{X}_G = X_G \cos\psi_3 + Y_G \sin\psi_3,\\ \bar{Y}_G = -Y_G \sin\psi_3 + Y_G \sin\psi_3,\\ \bar{Z}_G = Z_G\end{gathered} \tag{1}$$

where $\bar{X}_G$, $\bar{Y}_G, \bar{Z}_G$ are the coordinates of the origin of the circular arc radius of curvature $\boldsymbol{G}$ in the mobile coordinate system $O\bar{X}\bar{Y}\bar{Z}$, $\psi_3 = \psi_i/i$ $-$ the rotation angle of the central wheel, $i-$ the transmission ratio of the precessional transmission [5].

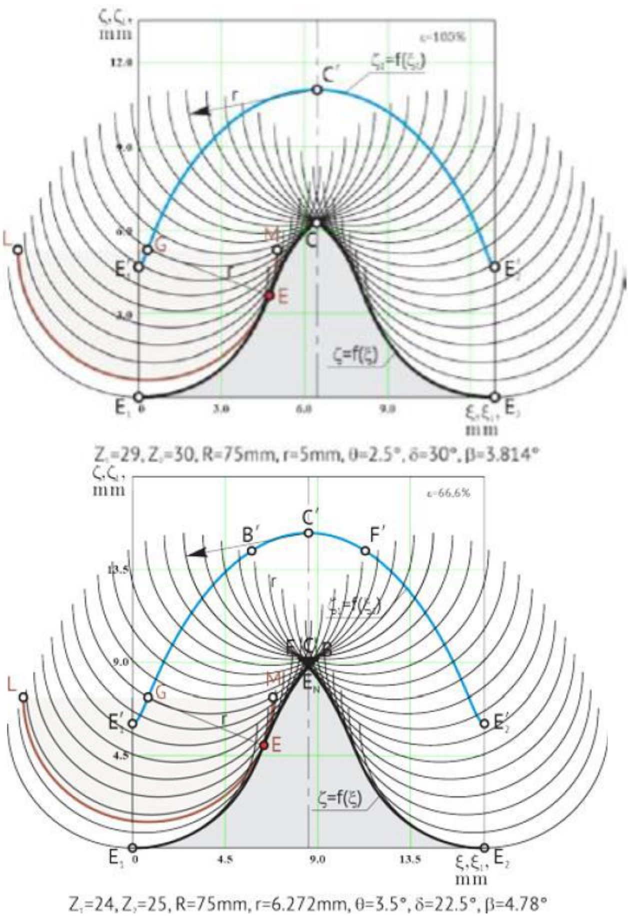


Fig. 1. Central wheel tooth profile described by the wrap of the family of circle arcs in the mesh with reference face contact ratio $\epsilon = 100\%$ (left picture) and $\epsilon < 66.6\%$ (right picture).

Equations (1) represent the developed trajectory of the motion of the origin G and the radius of the circular arc r on the sphere with radius R.
After that will represent the working surface of the flanks of the satellite teeth with a circular arc profile in conical form with the generator extensions intersecting at the precession center O (in normal section) in the mobile coordinate system $O\bar{X}\bar{Y}\bar{Z}$, using the condition from differential geometry:

$$re = r\cos\beta \ \ or \ X\bar{X}_G + Y\bar{Y}_G + Z\bar{Z}_G = Rr\cos\beta, \qquad (2)$$

where e is the unit vector along the axis of the toothed cone with a circular arc profile, $\beta -$ the apex angle of the conical surface of the teeth with a circular arc profile, [6].
The equation of the envelope of the family of circular arcs LEM on the sphere with radius R by simultaneously solving the equations that describe the envelope of the family of flank surfaces of the satellite teeth with a circular arc profile:

$$\phi(X, Y, Z, \psi) = XX_G + YY_G + ZZ_G - Rr\cos\beta = 0$$
$$\frac{d\phi}{d\psi}(X, Y, Z, \psi) = 0 \qquad (3)$$

and the equation of the sphere's surface:

$$X^2 + Y^2 + Z^2 - R^2 = 0. \qquad (4)$$

For this, from equations (3) and (4) will write:

$$\frac{d\phi}{d\psi} = X\frac{\partial\bar{X}_G}{\partial\psi} + Y\frac{\partial\bar{Y}_G}{\partial\psi} + Z\frac{\partial\bar{Z}_G}{\partial\psi},$$
$$\frac{\partial\bar{X}_G}{\partial\psi} = \frac{\partial X_G}{\partial\psi}\cos\psi_3 - \frac{X_G}{u}\sin\psi_3 + \frac{\partial\bar{Y}_G}{\partial\psi}\sin\psi_3 + \frac{Y_G}{u}\cos\psi_3,$$
$$\frac{\partial\bar{Y}_G}{\partial\psi} = -\frac{\partial X_G}{\partial\psi} sin\,\psi_3 - \frac{X_G}{u} cos\,\psi_3 + \frac{\partial Y_G}{\partial\psi} cos\,\psi_3 - \frac{Y_G}{u} sin\,\psi_3, \qquad (5)$$
$$\frac{\partial\bar{Z}_G}{\partial\psi} = \frac{\partial Z_G}{\partial\psi},$$
$$\frac{\partial X_G}{\partial\psi} = -R\cos\delta\,(1-\cos\theta)\cos 2\psi - R\sin\delta\sin\theta\cos\psi,$$
$$\frac{\partial Y_G}{\partial\psi} = -R\cos\delta\,(1-\cos\theta)\cos 2\psi - R\sin\delta\sin\theta\cos\psi,$$
$$\frac{\partial Z_G}{\partial\psi} = R\,cos\,\delta\,sin\,\theta\,sin\,\psi$$

After substituting (5) into (3) and (4), will obtain the envelope equations on the sphere:

$$X_i = \frac{-(ab+de)\pm\sqrt{(ab+de)^2+(1+a^2+d^2)(R^2-b^2-e^2)}}{1+a^2+d^2}, \qquad (6)$$
$$Y_i = aX_i + b,$$
$$Z_i = dX_i + e,$$

where:

$$a = \left(\bar{X}_G\frac{\partial\bar{Z}_G}{\partial\psi} - \bar{Z}_g\frac{\partial\bar{X}_G}{\partial\psi}\right) \Big/ \left(\bar{Z}_G\frac{\partial\bar{Y}_G}{\partial\psi} - \bar{Y}_g\frac{\partial\bar{Z}_G}{\partial\psi}\right),$$

$$b = \left(-R^2 \cos\beta \frac{\partial \bar{Z}_G}{\partial\psi}\right) \Big/ \left(\bar{Z}_G \frac{\partial \bar{Y}_G}{\partial\psi} - \bar{Y}_g \frac{\partial \bar{Z}_G}{\partial\psi}\right),$$
$$d = -\frac{(\bar{X}_G + a\bar{Y}_G)}{\bar{Z}_G},$$
$$e = \frac{R^2 \cos\beta - b\bar{Y}_G}{\bar{Z}_G}$$

Equations (6) represent the wrap of the family of circular arcs on the sphere and the profile of the central wheel teeth. To identify the profile of the central wheel teeth, in the normal section, will project the envelope from the sphere onto a plane P perpendicular to two directrices passing through two similar and adjacent points on the teeth profile on the sphere, for example through E'_1 and E'_2 respectively (fig.1).

The coordinates E'_1 and E'_2 can be expressed by:

$$\begin{aligned} X_{E_1} &= X_1 = X_i|_{\psi=0}, \\ Y_{E_1} &= Y_1 = Y_i|_{\psi=0=-R\cos(\delta+\theta+\beta)}, \\ Z_{E_1} &= Z_1 = Z_i|_{\psi=0=-R\cos(\delta+\theta+\beta)}, \\ X_2 &= X_2 = X_i|_{\psi=\frac{2\pi Z_2}{Z_1}}, \\ Y_{E_2} &= Y_2 = Y_i|_{\psi=\frac{2\pi Z_2}{Z_1}}, \\ Z_{E_2} &= Z_2 = Y_i|_{\psi=\frac{2\pi Z_2}{Z_1}} \end{aligned} \tag{7}$$

To project the envelope from the sphere, through points E_1 and E_2, will draw a plane P_1 perpendicular to the directrices O_{E1} and O_{E2} [7]. The equation of this plane can be:

$$[E_1E_2 \times E_1E][OE_1 \times OE_2] = 0 \tag{8}$$

where E is an arbitrary point on plane P_1.

Equation (8) is represented in the form:

$$\begin{vmatrix} i & j & k \\ X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X - X_1 & Y - Y_1 & Z - Z_1 \end{vmatrix} \begin{vmatrix} i & j & k \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0 \tag{9}$$

or

$$A_1X + B_2Y + C_1Z + D_1 = 0 \tag{10}$$

where:

$$\begin{aligned} A_1 &= (Z_2 - Z_1)(X_2Z_1 - X_1Z_2) - (Y_2 - Y_1)(X_1Y_2 - X_2Y_1), \\ B_1 &= (X_2 - X_1)(X_1Y_2 - X_2Y_1) - (Z_2 - Z_1)(Y_1Z_2 - Z_1Y_2), \\ C_1 &= (Y_2 - Y_1)(Y_1Z_2 - Z_1Y_2) - (X_2 - X_1)(Z_1X_2 - X_1Z_2), \\ D_1 &= -A_1X_1 - B_1Y_1 - C_1Z_1. \end{aligned} \tag{11}$$

Next, using the rules of spherical trigonometry, will project the envelope expressed by equations (6) from the sphere onto plane P_1, described by equation (10). For this, will identify the place of the intersection points of the family of lines passing through the precession center O and the points that form the envelope on the sphere [8].

The equation of the line passing through the precession center O and an arbitrary point E on the envelope (6) has the form:

$$Y = X\frac{Y_i}{X_i},\ Z = X\frac{Z_i}{X_i} \qquad (12)$$

Solving equations (10) and (11) simultaneously, will obtain the equations of the envelope of the family of circular arcs (10) in the $OXYZ$ coordinate system.

$$X_{iP} = -\frac{D_1X_i}{A_1X_i+B_1X_i+C_1Z_i}, Y_{iP} = X\frac{Y_i}{X_i}, Z_{iP} = X\frac{Z_i}{X_i}. \qquad (13)$$

For the 2D representation of the envelope of the family of circular arcs, will transit from presenting the envelope in X, Y, Z coordinates in the $OXYZ$ coordinate system to its equation in ξ and ζ coordinates in the $E_1\xi\zeta$ coordinate system, located in plane P_1 (10), according to the expressions:

$$E_1E_2 = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2},$$
$$E_1E = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z_2 - Z_1)^2} = \sqrt{\xi^2 + \zeta^2},$$
$$E_2E = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2} = \sqrt{(E_1E_2 - \xi)^2 + \zeta^2}, \qquad (14)$$

where E is an arbitrary point on plane P_1 [9].

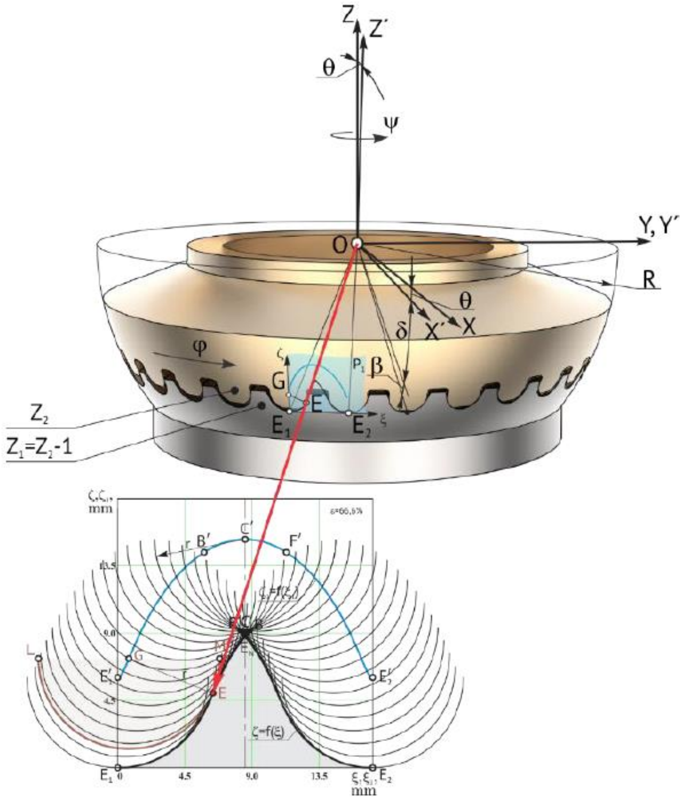


Fig. 2. Description of the central wheel tooth profile by the envelope of the family of circular arcs in the mobile coordinate system $O\bar{X}\bar{Y}\bar{Z}$.

Solving (14), will obtain the equations of the envelope of the family of circular arcs (fig. 2), projected in 2D onto plane P_1, in ξ and ζ coordinates. The function $\xi = f(\zeta)$ expresses the envelope of the circular arc satellite tooth profiles and represents the profile of the central wheel teeth in normal section, defined by the expressions:

$$\xi = \{(E_1E_2)^2 + (X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2 - (X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2\}(2E_1E_2)^{-1},$$

$$\zeta = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2 - \xi^2}. \qquad (15)$$

The contact point E of the satellite wheel teeth in a circle arc and the central wheel, for any angular position $\boldsymbol{\psi}$ of the crankshaft, is located at the distance of the radius *r* of the circle arc, on the normal at the velocity vector $\boldsymbol{V_G}$ of its origin at the point ***G,*** fig. 2 [10].

The profile of the central wheel teeth in the normal section is designed in the CAD/CAE/CAM-Catia modeling system V5R19. The principle of forming the wrap of the circle arcs family described with equation (15) with the placement of their radii origins [11] on the curve described with equations (1), represents the kinematic model of the processes $G_{r.s}$ of generation through spatial rolling of the central wheel teeth, with the reproduction of the geometry and kinematics of the interaction of the teeth from the real *PPT*.

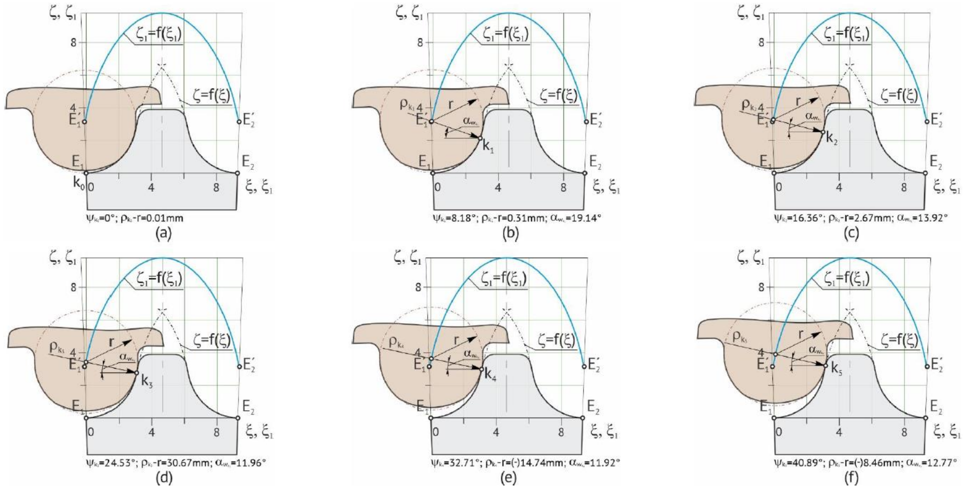


Fig. 3. The conjugation succession of the flank profiles of one and the same pair of teeth in contacts ki with characteristics $\rho_{k\mathrm{i}} - r$ and α_{wki} as a function of the precession angle ψ_H in the gear A^D_{CX-CV}: $Z_1 = 45, Z_2 = 46, R = 75mm, r = 3.14mm, \theta = 2.5°, \delta = 22.5°, \beta = 2.4°$.

From the analysis of fig. 1, was find the convex-convex and convex-rectilinear contact are characteristic of the flank's conjugation with the trip area of the central wheel teeth [12]. Using this geometric aspect, it is possible to change the shape of the tooth, implicitly of the performance characteristics of the contact, by shortening its height to the level that would provide only one convex-concave contact (fig. 3). By changing the shape of the central wheel tooth by shortening its height (fig.4, c, d) the teeth flanks conjugate in convex-concave contact at the limit at point k_i

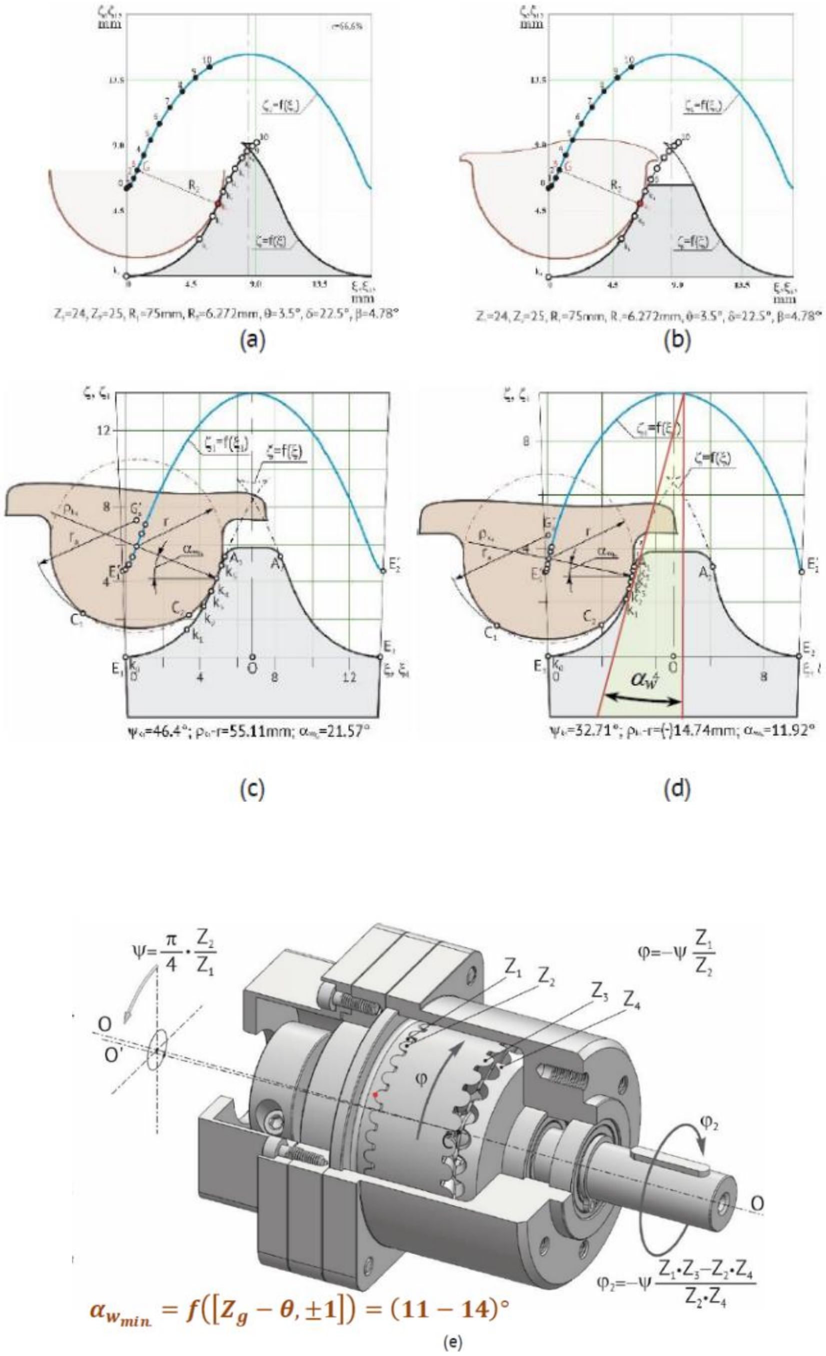
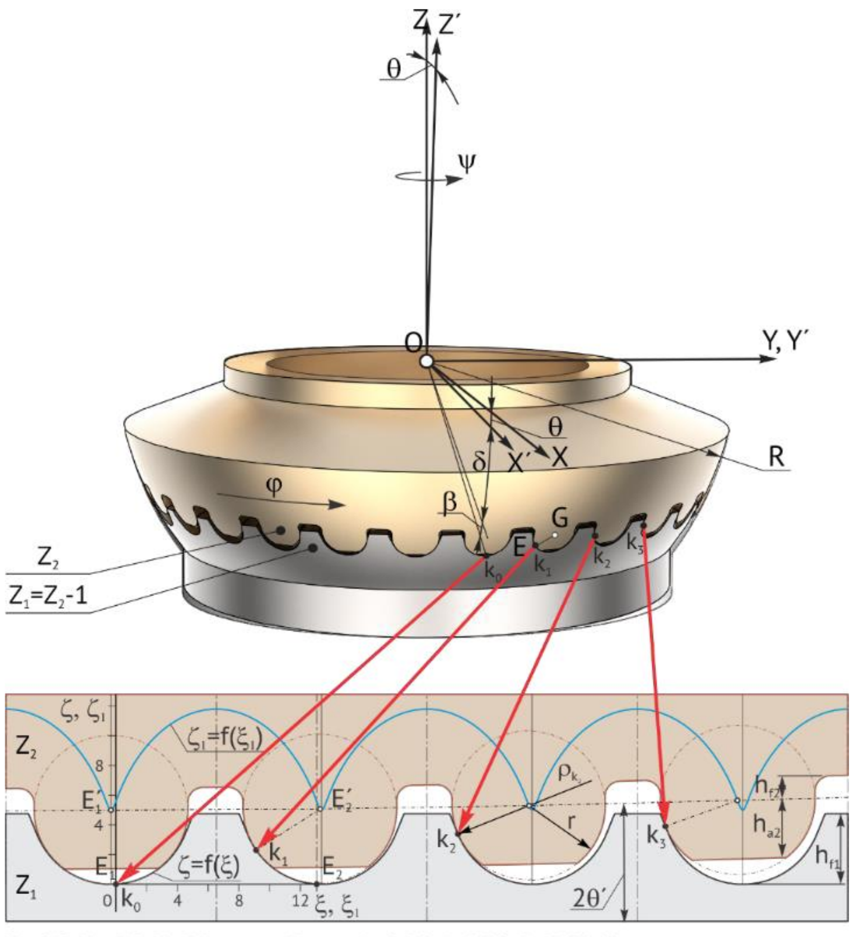


Fig. 4 Tooth shape modification a function of contact dispersion k_i: k_i contact points transposed onto the unmodified (a) and modified (b) central wheel tooth profile;
(c), (d) - dispersion of contact points for one and the same pair of teeth as a function of ψ ;
(e) - view of the multi-pair mesh A_{CX-CV}^{D} for the same ψ.

(fig.5) and in the area between it and the type of modified tooth, the flanks conjugate in convex-rectilinear contact. Therefore, depending on the modified height of the central wheel teeth and parametric configuration [Z_g - θ, ±*1*], which would ensure the transformation of [13] the motion with constant transmission ratio, it's possible to provide single, bipartite, tripartite gearing etc. it's possible to intervene on the multiplicity of the gearing of the contact geometry and the kinetic-static of the teeth contact. Based on computer simulations on virtual models, it was found that at the variation of crankshaft precession angle $0^o < \psi < 37^o$, the convex-concave contact is provided in the pairs of teeth conjugated in the contacts k_0, k_1, k_2 and k_3, shown in the unfolding of the teeth profiler [14] in the fig. 5. Thus, for gear with geometric parameters Z_1=29, Z_2=30, *R=75, r = 5 mm*, $\theta = 2.5°, \delta = 30°, \beta = 3.8°$.



Z_1=29, Z_2=30, R=75mm, r=5mm, θ=2.5°, δ=30°, β=3.814°

Fig. 5. 2K-H precessional gear with small difference of curvature in the contact of geared teeth with the multiplicity e=27,5 %.

Figure 6 presents dispersion of contact points k_i, at the same angular coordinate ψ of the crankshaft [15] in the gear A^D_{CX-CV}. Also, CAD model of A^D_{CX-CV} precessional planetary transmission,[16].

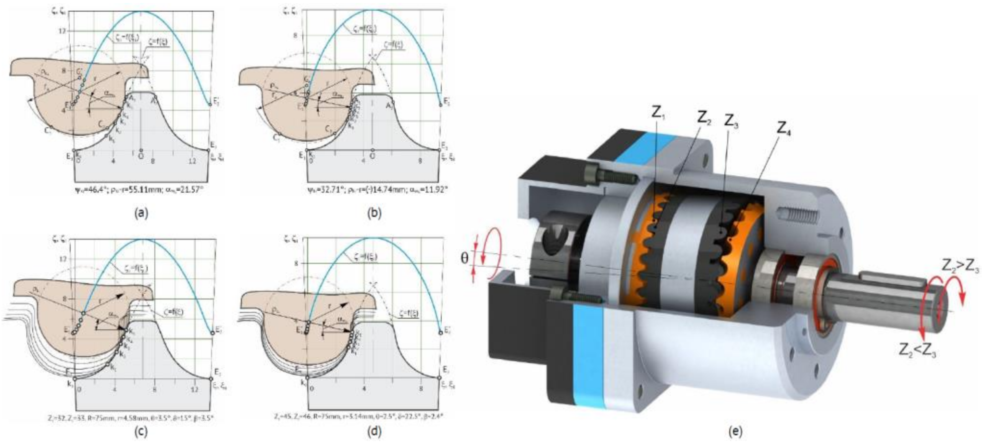


Fig. 6. Dispersion of contact points k_i, pressure angle $\alpha_w$and positional succession of multi-pair conjugated tooth profiles in contacts $k_0 \dots k_j$ at the same angular coordinate ψ of the crankshaft in the gear A^D_{CX-CV}: (a), (c) $Z_1 = 32, Z_2 = 33, R = 75mm, r = 4.587mm, \theta = 3.5°, \delta = 15°, \beta = 3.5°$; (b), (d) $Z_1 = 45, Z_2 = 46, R = 75mm, r = 3.14mm, \theta = 2.5°, \delta = 22.5°, \beta = 2.4°$ and kinematic particularities of the transmission (e).

3. Conclusions

For the proper functioning of the A^D_{CX-CV} gear firstly must be identified the geometry of the contacts and secondly mathematical arguments of the contact geometry. For the tooth profile description of the central wheel is necessary to draw circular arcs with radius EG -red line, fig.1. The coordinates are transferred of the origin of the circular arc radius $\boldsymbol{G}$ into the mobile coordinate system $O\bar{X}\bar{Y}\bar{Z}$ linked, to the central wheel according to the relation (1).
Equations (6) represent the wrap of the family of circular arcs on the sphere and the profile of the central wheel teeth. To identify the profile of the central wheel teeth, in the normal section, will project the envelope from the sphere onto a plane P perpendicular to two directrices passing through two similar and adjacent points on the teeth profile on the sphere, for example through E'_1 and E'_2 respectively (fig.1). The contact point E of the satellite wheel teeth in a circle arc and the central wheel, for any angular position $\boldsymbol{\psi}$ of the crankshaft, is located at the distance of the radius r of the circle arc, on the normal at the velocity vector $\boldsymbol{V_G}$ of its origin at the point $\boldsymbol{G}$**,** fig. 2.
The profile of the central wheel teeth in the normal section is designed in the CAD/CAE/CAM-Catia modeling system V5R19. The principle of forming the wrap of the circle arcs family described with equation (15) with the placement of their radii origins on the curve described with equations (1), represents the kinematic model of the processes $G_{r.s}$ of generation through spatial rolling of the

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