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The influence of Poisson's ratio in the calculus of functionally graded plates

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Abstract. A current problem, of increased difficulty, which is still under continuous development, is the calculus of functionally graded plates (FGPs). Functionally graded materials (FGMs) represent a special category of composite materials, usually made on the basis of two materials, with very different properties, so they vary continuously between the extreme surfaces of the material, where the properties are those of the respective materials in their pure state. Today, the most used materials used in the construction of FGMs are ceramic materials and metals. Their volume fractions, in the thickness direction, varies continuously, according to a material law, valid for all material properties. A problem, generally poorly substantiated, is the one related to the assumption of a constant value of the Poisson's ratio over the entire plate thickness of the functionally graded plates. This hypothesis admits an analytical solution, by direct integration, of the stiffness of the plate, but it certainly does not reflect reality. There are some ways of approaching the calculation of functionally graded plates, such as the multilayer plate concept or the equivalent plate concept, which can take into account the variation of the Poisson's ratio on the plate thickness. The paper highlights this aspect and, in addition, evaluates the influence of the variation of Poisson's ratio on the calculation of displacements, stresses and natural vibrations of functionally graded plates. The work represents both an original way of approaching the calculation of functionally graded plates and a quantitative substantiation of the hypothesis of a Poisson coefficient with a constant value.

Keywords: functionally graded plates (FGPs), material law, power coefficient.

1. Introduction

A new category of materials has appeared belonging to the broad class of composite materials, which are attracting more and more attention to specialists

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[1], [2]. It's about of Functionally Graded Materials (FGMs) [3], [4]. The idea of a smooth variation of material properties came from Japan in the 80's. This category of materials is based on two materials, with very different characteristics, whose the volume fractions in the thickness direction varies continuously, according to a certain law $F(z)$ - Figure 1. This function can be for any of the material properties: Young's modulus, shear modulus, Poisson's ratio, density, etc. On the extreme faces we find the two materials in total proportions, with the specific characteristics of those two materials.

The technical literature in the field presents several material laws: the power law, the Reuss Law (RL), the Voigt Law (VL), the Law of the Representative Volume Element (LRVE), the Mori-Tanaka Law (MTL), the Exponential Law (EL), the Sigmoid Law (SL) and others [4], [5]. FGMs are becoming more widely used in various fields, from biomechanics to aerospace structure [5], [6].

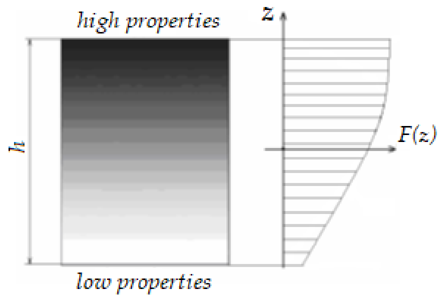

Fig. 1. Model of building functional graded material.

The interest of specialists for such materials is great and both the manufacture of such materials and the development of calculus are also known to be of particular interest. The results presented in this paper refer only to the Power Law, this being one of the most used. The methods, methodologies and conclusions presented in this paper are valid for any of the material laws mentioned above.

2. Material Power Law

The coordinate system, adopted in our study of FGPs is presented in the Figure 2 and the mathematical expression of the material law is relation (1).

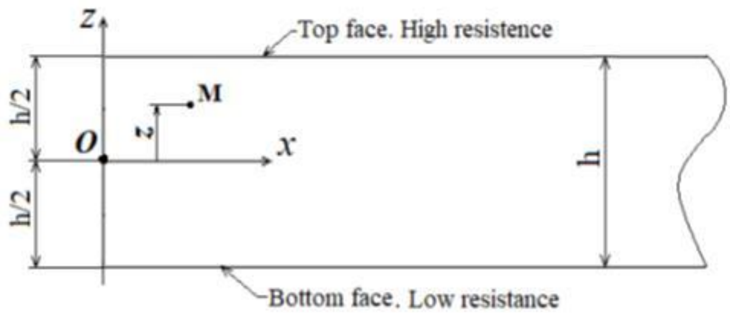

Fig. 2. The addoped coordinate system.

$$E(z) = E_b + \left(E_t - E_b\right)\left(\frac{1}{2} + \frac{z}{h}\right)^k \tag{1}$$

In the relation (1), z is the coordinate along the thickness plate (Figure 2), the indices t and b refer to the upper and the lower surface, respectively (Figure 1 and 2); h is the thickness of the plate (Figure 1 and 2) and k is the power coefficient, which may have different values, less or greater than 1. The same mathematical relation is used for any material properties. The evolution of Young modulus versus power coefficient k is presented in the Figure 3. According to the same material law, the Figure 4 shows the variation of the Poisson's ratio along the plate thickness, for several values of the power coefficient k .

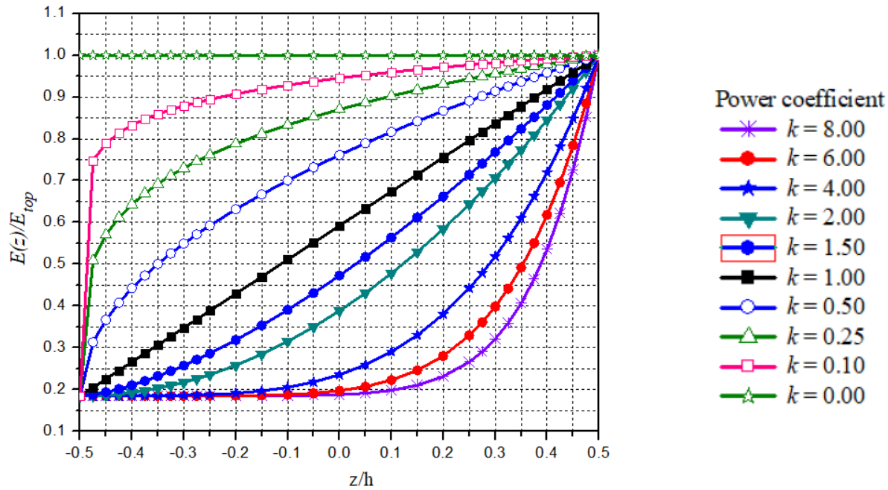

Fig. 3. Material property variation for different values of power coefficient k .

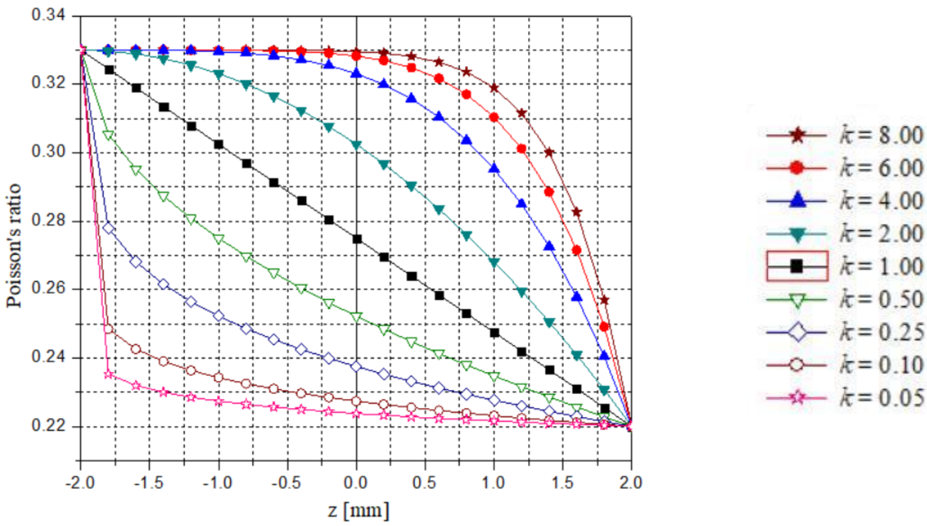

Fig. 4. Poisson’s ratio variation along the plate thickness.

3. Case study

A plane circular plate is loaded with an uniformly distributed load, as the Figure 5 shows, having the radius R of 50 [mm], the thickness h of 4 [mm] and the distributed load p of 5 [MPa]. The material properties of the FGP are presented in the Table 1.

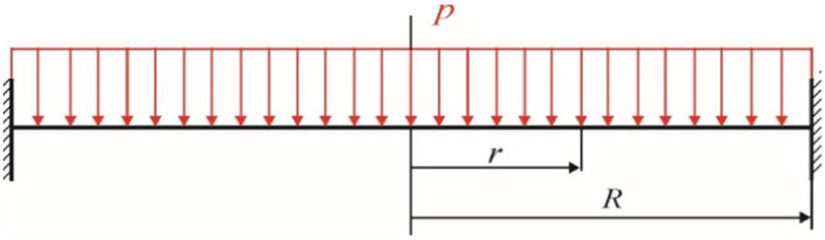

Fig. 5. Functionally graded circular plane plate.

Table 1. Material properties

Materials:	Ceramic Al_2O_3	Aluminum
Positions:	Top	Bottom
E [Pa]	$3.8 \cdot 10^{11}$	$7.0 \cdot 10^{10}$
ρ [kg/m3]	3960	2700
ν [-]	0.22	0.33

When the variation of the Poisson's ratio along the thickness is neglected, the bending rigidity $D \equiv D_a$ is analytically calculated [7], [8] with the relation (2):

$$D = D_a = \frac{1}{1-\nu^2} \int_{-h/2}^{h/2} E(z) \cdot z^2 \cdot \mathrm{dz} \tag{2}$$

The result is:

$$D = D_a = \frac{E_b h^3}{12(1-\nu^2)} + \frac{(E_c - E_b)h^3}{1-\nu^2} \cdot \left[\frac{1}{3+k} - \frac{1}{2+k} + \frac{1}{4(k+1)} \right] \tag{3}$$

When the concept of multilayer plate is used, with the calculus model presented in the Figure 5, the bending rigidity $D \equiv D^*$ is calculated [11] with relations (4)...(7).

$$D^* = \frac{A \cdot C - B^2}{A} \tag{4}$$

$$A = \sum_k \frac{E_k}{1-\upsilon_k^2} \cdot \left(z_k - z_{k-1} \right) \tag{5}$$

$$B = \sum_k \frac{E_k}{1-\upsilon_k^2} \frac{z_k^2 - z_{k-1}^2}{2} \tag{6}$$

$$C = \sum_k \frac{E_k}{1-\upsilon_k^2} \cdot \frac{z_k^3 - z_{k-1}^3}{3} \quad (7)$$

The index k , in the relations (12)...(14) refers to the k^{th} layer, as in the Figure 6.

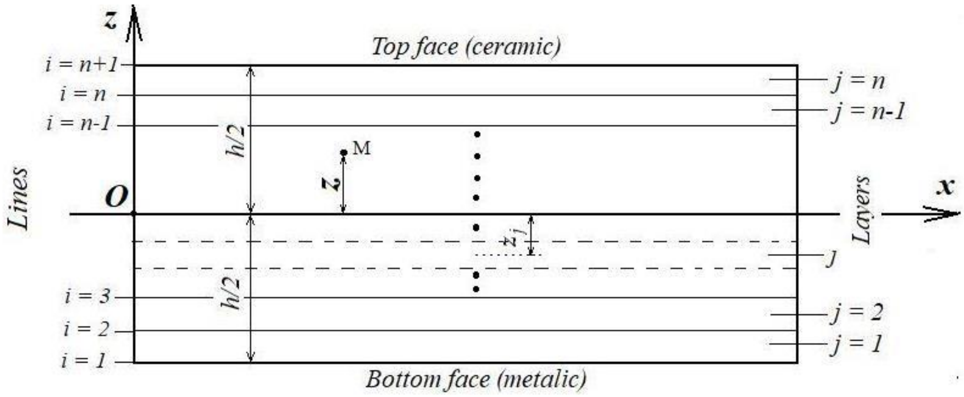

Fig. 6. The multilayer plate model.

Using the relations (3) and (4), for a constant Poisson's ratio of 0.30, leads to the following results for the bending rigidity of the FGP:

$$D^* = 2115.80 \ [\text{Pa} \cdot \text{m}^3] \quad (8)$$

$$D_a = 2114.30 \ [\text{Pa} \cdot \text{m}^3] \quad (9)$$

Between D^* and D_a , the error is only 0.07%. The variation of the material characteristics on the plate thickness, for the 20 considered layers, is reflected in Table 2, according to the same material law - the Power Law – for k = 0.05.

Table 2. The values of material characteristics on each layer

Layers of the plate thickness				
Layer No.	$z_{layer}^{av.}$ [m]	$E(z_{layer}^{av.})$ [MPa]	$\nu(z_{layer}^{av.})$ [-]	$\rho(z_{layer}^{av.})$ [kg/m^3]
1	-0.0019	327785.62	0.23853	3747.774
2	-0.0017	342342.08	0.23336	3806.939
3	-0.0015	349387.64	0.23086	3835.576
4	-0.0013	354127.71	0.22918	3854.842
5	-0.0011	357720.51	0.22791	3869.445
6	-0.0009	360621.89	0.22688	3881.237
7	-0.0007	363059.54	0.22601	3891.145
8	-0.0005	365163.91	0.22526	3899.698
9	-0.0003	367016.89	0.22461	3907.230

10	-0.0001	368673.28	0.22402	3913.962
11	0.0001	370171.64	0.22349	3920.052
12	0.0003	371540.11	0.22300	3925.615
13	0.0005	372799.88	0.22256	3930.735
14	0.0007	373967.31	0.22214	3935.480
15	0.0009	375055.31	0.22176	3939.902
16	0.0011	376074.24	0.22139	3944.044
17	0.0013	377032.53	0.22105	3947.939
18	0.0015	377937.16	0.22073	3951.616
19	0.0017	378793.95	0.22043	3955.098
20	0.0019	379607.82	0.22014	3958.406
Layer average value		*365443.95*	*0.22517*	*3900.837*

Table 3 presents the values of bending rigidity for different values of Poisson's ratio, as follows (in Table 2): ceramic material (ν_{ceram}), average value of those two materials (ν_{med}), average values depending on power coefficient k of the layer values (ν), constant value of 0.30 (ν_{const}) and aluminum material (ν_{alum}).

Table 3. The analytical values of bending rigidity by direct integration

k	0.05	0.10	0.25	0.50	1.00	2.00	4.00	6.00
$\nu_{ceram.}$ [-]	0.220	0.220	0.220	0.220	0.220	0.220	0.220	0.220
D [Nm]	2021.87	1928.13	1710.98	1484.42	1261.03	1087.29	938.37	847.36
$\nu_{med.}$ [-]	0.275	0.275	0.275	0.275	0.275	0.275	0.275	0.275
D [Nm]	2081.42	1984.91	1761.37	1528.14	1298.17	1119.31	966.01	872.32
ν [-]	0.225	0.230	0.242	0.257	0.275	0.293	0.308	0.314
D [Nm]	2028.05	1934.83	1400.86	1412.09	1093.84	861.65	756.02	720.52
$\nu_{const.}$ [-]	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300
D [Nm]	2114.30	2016.27	1789.20	1552.28	1318.68	1137.00	981.27	886.10
$\nu_{alum.}$ [-]	0.330	0.330	0.330	0.330	0.330	0.330	0.330	0.330
D [Nm]	2159.14	2059.03	1827.15	1585.2	1346.65	1161.11	1002.1	904.89

The influence of Poisson's ratio in the calculus of FGPs is reflected by the values in Table 3 and the curves in Figure 7.

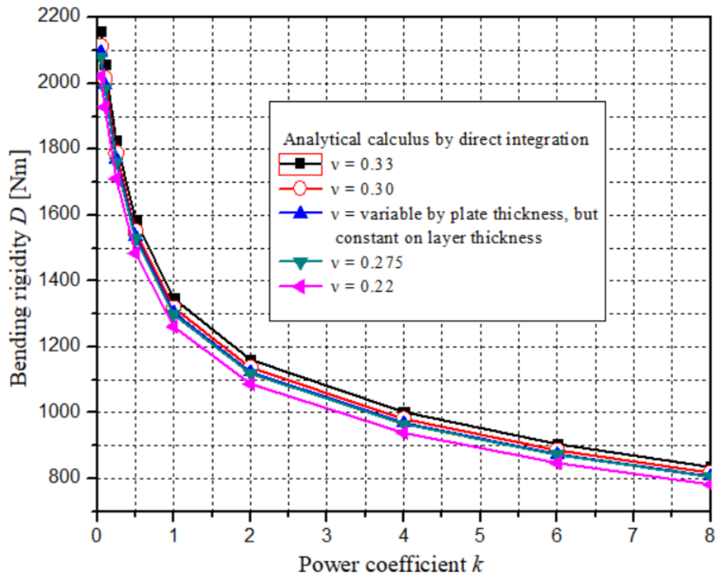

Fig. 7 Bending rigidity versus power coefficient k, for different Poisson's ratio.

The influence of Poisson's ratio on the bending rigidity is spread over all other parameters, such as efforts, stresses, natural frequencies, etc. Its influence on the maximum transverse displacement w_{max} is illustrated by the values presented in the Table 4 and calculated with relation (10). The errors are calculated in relation to the results corresponding to the 0.30 value of the Poisson's ratio (column 4 of Table 4).

$$w_{\max} = \frac{pR^4}{64D} \tag{10}$$

Table 4. The influence of ν on the $w_{\max}$ displacement

k	Constant, at the value of 0.05			
n	0.220	0.275	**0.300**	0.330
D [Nm]	2021.87	2081.42	**2114.3**	2159.14
w_{max} [m]	0.000241	0.000235	**0.000231**	0.000226
Er. [%]	***4.57***	***3.73***	***-***	***-2.08***
k	0.05	0.10	0.25	0.50
n	0.225	0.23	0.242	0.257
D [Nm]	2028.05	1934.83	1400.86	1412.09
w_{max} [m]	0.000241	0.000252	0.000349	0.000346
Er. [%]	***4.25***	***9.28***	***50.93***	***49.73***

The value of 0.30 is often used in FGPs calculus in the field literature. In Table 5, the influence of the Poisson's ratio on the maximum radial stress in the FGP is presented quantitatively for the case study. The relation (11) was used [11], [12],

$$\sigma_r = -z\frac{E_a}{1-\nu^2}\cdot\frac{pR^2}{8D_a} \tag{11}$$

where E_a results from the stiffness definition relation [12], [13]:

$$E_a = \frac{12(1-\nu^2)}{h^3}D_a \tag{12}$$

Table 5. The influence of ν on the maximum radial stress

k	0.05	0.10	0.25	0.50	1.00	2.00
n	0.225	0.23	0.242	0.257	0.275	0.293
D [Nm]	2028.05	1934.83	1400.86	1412.09	1093.84	861.65
E [MPa]	366901	354782	323417	283728	232750	181213
σ_r^ν [MPa]	**595.501**	**605.024**	**766.350**	**672.304**	**719.346**	**718.937**
σ_r [MPa]	**621.268**	**629.691**	**792.824**	**689.999**	**730.709**	**722.216**
Err. [%]	***-4.15***	***-3.92***	***-3.34***	***-2.56***	***-1.56***	***-0.45***

In Table 6, the influence of the Poisson's ratio on the first free vibration frequency of the FGP is presented quantitatively for the case study. The used relation is [9],

$$f_i = \frac{\alpha_i}{2\pi R^2}\sqrt{\frac{D}{\rho h}} \tag{12}$$

where α_i is a constant [9], [10] which depends on the number of nodal circles (s) and the number of nodal diameters (n), Figure 8. For the first frequency, α_i =0.21.

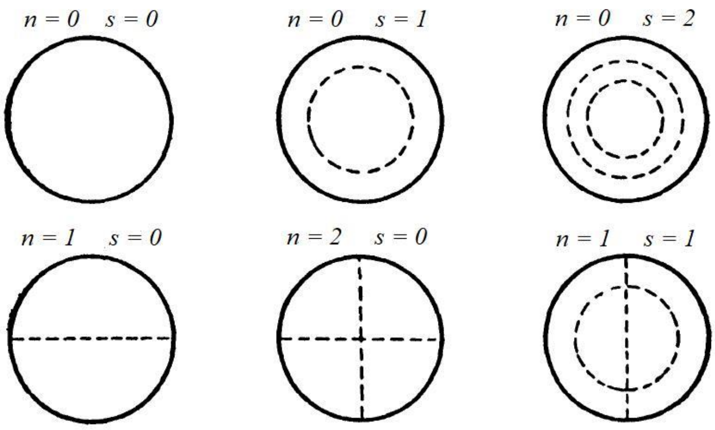

Fig. 8. Nodal cricles and nodal diameters.

Table 6. The influence of ν on the first free vibration frequency

k	0.05	0.10	0.25	0.50	1.00	2.00
ν	0.225	0.23	0.242	0.257	0.275	0.293
D [Nm]	2028.05	1934.83	1400.86	1412.09	1093.84	861.65
ρ [kg/m^3]	3849.293	3802.381	3680.971	3527.336	3330.000	3130.500
f_1^{ν} [Hz]	7463.54	7334.82	6343.26	6505.85	5893.20	5394.54
f_1 [Hz]	**7414.86**	**7242.44**	**6162.56**	**6187.21**	**5445.54**	**4833.14**
Err. [%]	***0.66***	***1.28***	***2.93***	***5.15***	***8.22***	***11.62***

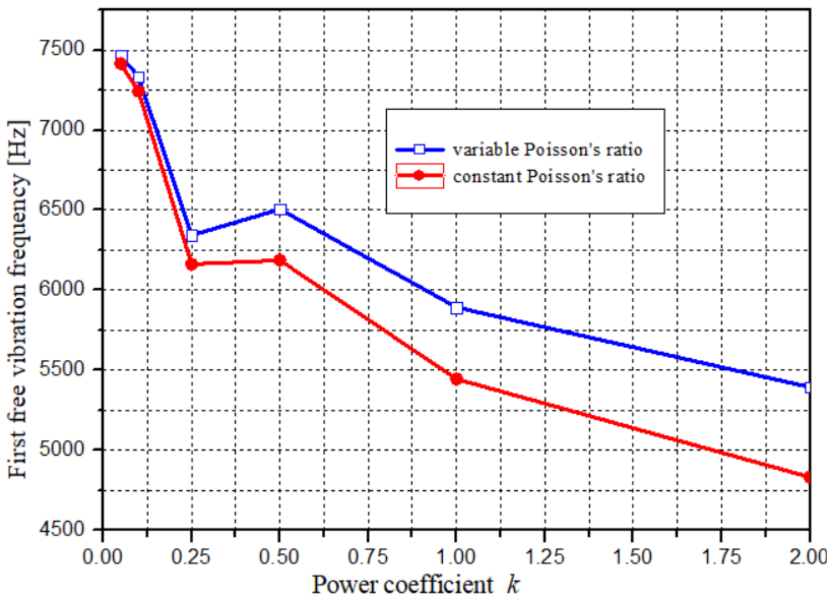

Fig. 9. First frequencies versus power coefficient k .

Figure 9, like the values in the tables above, shows a small influence of the Poisson's ratio at low values of the power coefficient k .

4. Conclusions

Poisson's ratio is one of the elastic characteristics of any material and consequently, characterizes its behavior.

As can be seen from the presented case study, Poisson's ratio influences any calculus regarding functionally graded plates.

The interest of the conducted research was to find the limits of its influence, that is, under what conditions the use of a constant value can replace its variable value, according to the function of the material, over the entire thickness of the plate.

Our research shows that for small values of the power coefficient k, using a constant value can be an acceptable solution. This fact allows an easier analytical calculus, but it is not the closest solution to the truth.

Taking into account the variation of the Poisson's ratio can be done in the numerical calculus, even without a dedicated software, using the concept of a multilayer plate or equivalent plate [7], [8].

The influence of Poisson's ratio must be further investigated for other categories of materials for the construction of FGPs and for other material laws.

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