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On signals processing on graphs, patterns, consensus and manifolds

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Abstract. The aim of this communication is to make several considerations regarding the way the notions of graph and Laplacian are involved in the concepts of signal processing on graphs, patterns, consensus and manifolds.
Keywords: signal processing on graphs, patterns, consensus, manifolds.

1. Signal processing on graphs

The motivation of this undertaking is to emphasize the role of the Laplacian properties in apparently unrelated areas associated to interactions represented by graphs. Recently the notion of signal processing on graph has been proposed as a generalization of that of signal [1].

First, we make a short survey regarding the way the Laplacian of a graph can be used in characterizing and processing signals associated to the graph.

Let us remind one of the fundamental principles in signal theory, i.e., that of decomposing signals with respect to a basis:

$$x = \sum_i \alpha_i \varphi_i$$

where φ_i are the elements of the basis and $\{\alpha_i\}$ represents the spectrum of x with respect to that basis. The ambiguity regarding the domain x and φ_i are defined on has been deliberately assumed since there are several possibilities time, space or both, including graph vertices.

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The notion of graph is well-known: it consists of a set of vertices, connected by edges characterized by weights. The concept of signal on a graph is intimately related to the above definition, the extra ingredient being that each note is considered to be "loaded" with a value associated to a vertex.
A sketch of a signal on a graph is shown in Fig. 1. Basically the graph is represented by vertices, edges with weights and "loads" for each vertex.

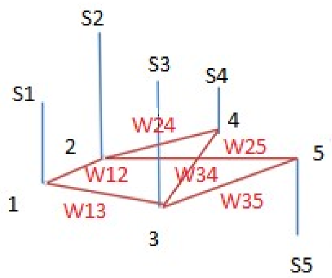

Fig. 1. Sketch of a signal on a graph.

A matrix related to the concept of graph is the Laplacian, defined for the un-weighted undirected graph as $L_G=D_G-A_G$, where D_G is the degree matrix and A_G the adjacencies one. For weighted graphs the Laplacian matrix has the form

$$L=\begin{pmatrix} \sum_k w_{1k} & -w_{12} & -w_{13} & \\ -w_{12} & \sum_k w_{2k} & & \\ & & & \\ & & & \sum_k w_{nk} \end{pmatrix},$$

where w_{ij} are the edge weights. Observe that in the standard definition of the Laplacian it is supposed that the diagonal entries of matrix A_G or the corresponding ones for the weighted case are null. This hypothesis is not compulsory if we accept vertices with self-edges. Such a situation appears when the graph models a circuit composed of identical grounded capacitors and a network of resistances connecting the "hot" terminals in the condition that positive or negative resistances are allowed across the capacitors. The consequence is that the zero eigenvalue corresponding to the constant eigenvector does not necessarily appear.
Coming back to signal processing on graph several basic challenges are: finding appropriate methods to determine the graph topologies and edges weights, developing transform methods and seeking to associate them with classical signal processing technique and, last but not least, find new specific ways to use the concept in a broader sense [2].
Part of responses to the above challenges are given in the same paper with reference to Fourier-graph spectral analysis analogies, filtering, translation,

modulation and down-sampling which are discussed in detail with emphasis on advantages, limitations and open problems.
The fundamental aspect that makes the problem of signal processing on graph appealing and more than interesting is the mathematical result which states that a symmetric matrix has real eigenvalues and orthogonal eigenvectors.
Moreover for a symmetric nxn matrix M the eigenvalues can be ascending ordered $\lambda_1 \le \lambda_2 \le \ldots \lambda_n$ with the corresponding eigenvectors ψ_i, I = 1, 2,…, n inferring the following relations:

$$\lambda_i = \min_{x \perp \Psi_1,\ldots,\Psi_{i-1}} \frac{x^T M x}{x^T x} \text{ and } \Psi_i = \arg \min_{x \perp \Psi_1,\ldots,\Psi_{i-1}} \frac{x^T M x}{x^T x}$$

which means that the i-th eigenvalue of M in ascending order can be obtained by minimizing the expression $\frac{x^T M x}{x^T x}$ for vectors x orthogonal on the first i-1 eigenvectors of M, the i-th eigenvector associated to the i-th eigenvalues being the vector x, orthogonal on the first i-1 eigenvectors which minimizes the same expression $\frac{x^T M x}{x^T x}$.
The above property allow making an analogy with Fourier transform by associating small eigenvalues corresponding to smooth eigenvectors with low frequencies and Fourier harmonic waveforms.

2. Patterns

The next aspect we mention regarding the concept of signal processing on graphs is pattern formation in nonhomogeneous cellular neural networks defined on graphs. The main idea is to study an architecture composed of identical grounded admittances connected to the vertices of a resistive grid.
The equations that describe the network are [3]:

$$Y(s)x_i(t) = \sum_{k \in N_i} A_k x_{i+k}(t) = -L_G x \tag{1}$$

where $Y(s)$ is the linear integro-differential operator corresponding to the admittances, N_i symbolize the neighborhood of each cell and L_G is the network Laplacian which acts on the vector x representing voltages on the graph nodes/vertices.
The equations can be solved by using the decoupling techniques which is based on the change of variable:

$$x_i(t) = \sum_{m=0}^{M-1} \Phi_M(i,m)\hat{x}_m(t) \tag{2}$$

where $\varphi_M(i,m)$ are the eigenvectors of the Laplacian which are known to be orthogonal and associated to real eigenvectors (positive for positive weights of the Laplacian).

Thus the spectrum of the discrete spatial and continuous time signal can be written:

$$\hat{x}_m(t) = \sum_{m=0}^{M-1} \Phi_M(m,i) x_i(t) \tag{3}$$

and after some manipulations the dynamics of the components of the initial conditions signal can be written as

$$Y(s)\hat{x}_m(t) = -K_A(m)\hat{x}_m(t) \tag{4}$$

or

$$Q(s)\hat{x}_m(t) + K_A(m)P(s)\hat{x}_m(t) = 0 \tag{5}$$

where $Q(s)$ and $P(s)$ are respectively the numerator and the denominator of the admittance $Y(s)$ and $K_A(m)$ the eigenvalues of the resistive network Laplacian the characteristic polynomial of each spatial mode amplitude being

$$Q(s) + K_A(m)P(s) = 0\,. \tag{6}$$

Since the standard Laplacian eigenvalues are nonnegative, it is apparent that, for real positive functions $Y(s)$ all modes are stable. However, if we accept negative resistances as well, either connected between nodes or between a node and ground the characteristic polynomial is no more Hurwitz and the dynamics can leads to an indefinite growth of the unstable modes. Indeed, since the interconnection matrix is symmetric, the eigenvalues will be again real and the eigenvectors orthogonal.

It seems that for first order cells it is not possible to have a "band" of unstable modes as in the case of double grid CNN or for other models. The possibility of having un unstable band of modes might exist for more complicated forms of $Y(s)$.

Last but not least, let us observe that, using the dynamics of a CNN defined on graph it is possible to make signal processing determined by the values of the eigenvalues of the graph which will influence the decay or growths of the modes in the system and the time the system is frozen to get the new spatial signal.

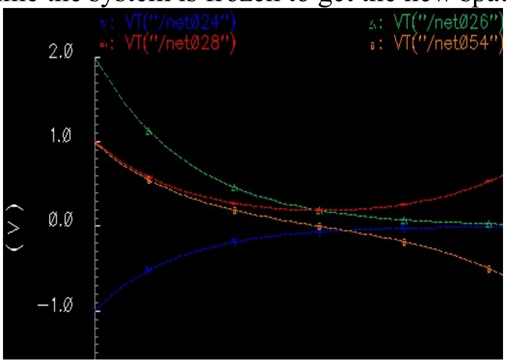

Fig. 2. State evolution leading to a pattern.

An example from [3] in Fig.1 shows the evolution of the node voltages for a graph defined cellular neural network exhibiting unstable modes.

3. Consensus protocol

Another topic related to the concept of signal processing on graphs is that of consensus protocol. There is a striking analogy between the dynamics of a nonhomogeneous CNN defined on a graph with Y(s)=s and the *linear weighted consensus protocol* over a weighted and undirected graph on |V| = n nodes, described by the graph G = (V,E,W), and the equation

$$\mathrm{d}x(t)/\mathrm{d}t = -L(G)x(t). \qquad (7)$$

In [4] the idea of using negative weights edges is analyzed with respect to the so called consensus protocol. The fundamental aspect is that of making a difference between synchronization and cluster synchronization.
In terms of graph circuit intuition, the difference consists in the fact that, for one or several negative weight edges in the graph, the above mentioned system of differential equations can still have a stable solution corresponding to global synchronization while for the case when the negative weight edges have value such that $s = 0$ is a natural frequency or root of the characteristic polynomial, the behavior of the system corresponds to cluster synchronization.
The intuition behind this behavior is that of having a root of the characteristic polynomial for a non-constant eigenvector as in the general case.
In other words the equilibrium point for $s = 0$ is no more the case all vertices have the same value/voltage but when the vertices exhibit a pattern proportional to the eigenvector associated to the $s = 0$ eigenvalue.

4. Manifolds

The last concept associated to graphs, eigenvalues and eigenvectors is that of manifold dimension reduction a concept apparently difficult to be considered as belonging to the topics of signal processing on graphs. The main idea of manifold learning and classification is that of finding a mapping between points in a high dimensional space and points on a manifold of (much) lower dimension with the constrain that points that are close in the initial space map to points that are also close on the manifold with respect to the geodesic distance.
According to [5] the optimum mapping that conserves distances is on the manifold consists of projections of the points in the initial space on the first *k* eigenvalues associated to the first (smallest) eigenvalues of the Laplacian of the data vectors. The main aspect that should be addressed is that of finding the weights for the underlying graph a matter that can be solved either using the heat transfer kernel

$$w_{ij} = e^{-\frac{\|x_i - x_j\|^2}{\sigma}} \quad (8)$$

or a 1/0 approach for signals within a prescribed neighborhood with respect to Euclidean distance. Thus, the solution of the problem is given by finding the eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \lambda_n$ with the corresponding eigenvectors ψ_i, i=1, 2,…, n, as above and choosing a dimension k for the manifold which consists of the projection of all points in the initial space to the k eigenvectors associated to the smallest (nonzero) eigenvalues.

A first demonstrative example is presented in Fig. 3 with projections on a 2D manifold (Fig. 4) and 1D manifold (Fig. 5) for 4 neighbours and σ = 10 for the heat transfer kernel.

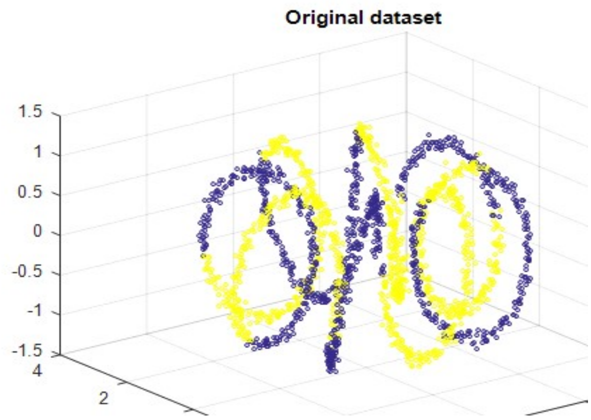

Fig. 3. Set of vectors belonging to two classes (1st example)

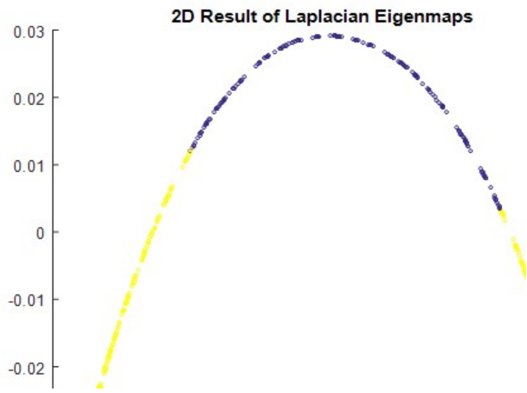

Fig. 4. Result of projection on a 2D manifold (N =4 sigma = 10)

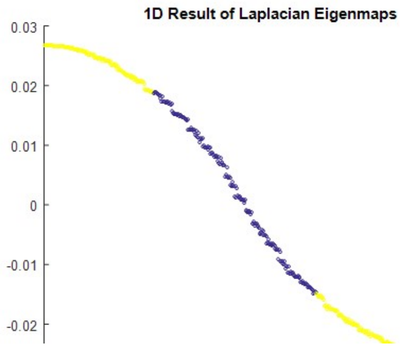

Fig. 5. Result of projection on a 1D manifold (N = 4 sigma = 10).

Another example corresponding to a manifold containing several classes of vectors is presented in Fig. 6 together with projections on a 2D (Fig.7) and 1D (Fig. 8) manifold for 10 neighbours and sigma = 10.

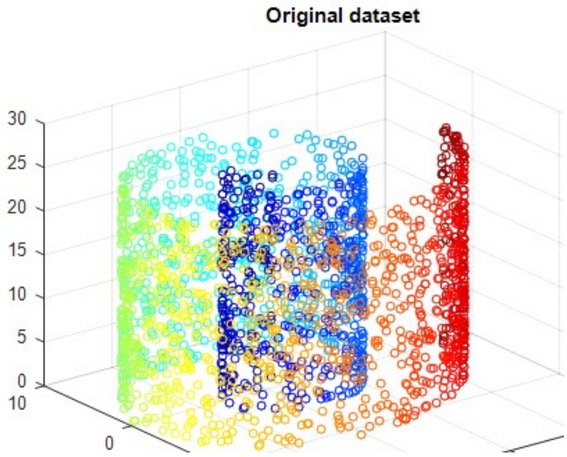

Fig. 6. Set of vectors belonging to several classes (2-nd example).

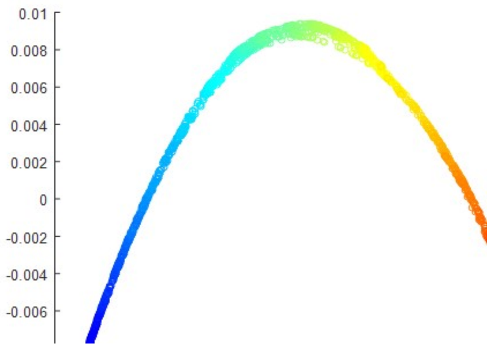

Fig. 7. Result of projection on a 2D manifold (N = 10 sigma = 10).

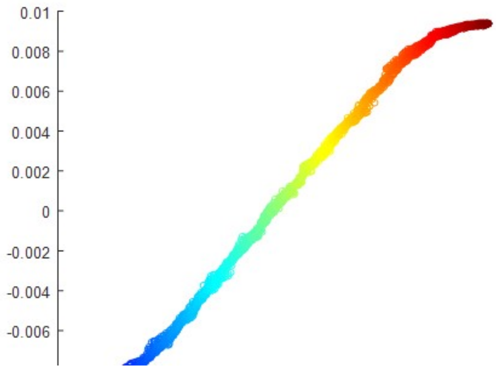

Fig. 8. Result of projection on a 2D manifold (N = 10 sigma = 10).

The above examples show that both N, the dimensions of the neighborhood as well as sigma are both important and rules to choose them for best classification results are not straightforward.

5. Concluding remarks

Connections between signal (processing) on graphs, patterns, consensus and manifolds have been shortly discussed, emphasizing the fact that all notions are related through the concept of graphs Laplacian, its real eigenvalues and orthogonal eigenvectors. It is expected that the observations made in this communications will help getting a better intuition on apparently unrelated areas.

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