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Perron-Frobenius theory applied to Leontief economic systems

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Abstract. Perron-Frobenius theory, well-known for non-negative matrices, is applied to the study of a category of economic systems described by Leontief-type models. For the considered models, the paper presents a qualitative point of view, oriented towards the solution existence / non-existence, as well as a quantitative point of view, oriented towards the solution construction. The main instruments used by our study are the structure of communication classes and the eigenvalue-eigenvector structure (also called matrix eigenstructure). A numerical case study is used for the illustration of theoretical concepts and results.
Key words: economical systems, systems with autonomous operation, Leontief models, linear algebra, non-negative matrices, Perron-Frobenius theory, matrix inequalities.

1. Introduction

The algebraic theory developed by Perron and Frobenius for non-negative matrices is applied to study economical systems with autonomous structure. To this end, the current section introduces the necessary mathematical notations (subsection 1), discusses the category of envisaged economic models (subsection 2), and provides an overview for the organization of the whole text (subsection 3).

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1.1. Notations for the use of Perron-Frobenius theory

The notation $X = [x_{ij}] \in R^{n\times m}$ designates a real matrix (in particular, a vector). The notation $\boldsymbol{X} \geq 0$, meaning $x_{ij} \geq 0$, $i = 1,\dots,n$, $j = 1,\dots,m$, denotes a *non-negative* matrix. The notation $\boldsymbol{X} > 0$, meaning $\boldsymbol{X} \geq 0$ and $\boldsymbol{X} \neq 0$, denotes a *semi-positive* matrix. The notation $X >> 0$, meaning $x_{ij} > 0$, $i = 1,\dots,n$, $j = 1,\dots,m$, denotes a *positive* matrix. The notation $\boldsymbol{X}^{\mathrm{t}}$ denotes the transposition of matrix $\boldsymbol{X}$. For a vector $x \in R^n$, we also use the equivalent writing $x \in R_+^n \Leftrightarrow x \geq 0$, and $x \in \mathrm{Int}(R_+^n) \Leftrightarrow x >> 0$, respectively. For two matrices, $\boldsymbol{X} = [x_{ij}]$, $Y = [y_{ij}] \in R^{n\times m}$, the notations $\boldsymbol{X} \geq \boldsymbol{Y}$, $\boldsymbol{X} > \boldsymbol{Y}$ and $X >> Y$ mean $\boldsymbol{X} - \boldsymbol{Y} \geq 0$, $\boldsymbol{X} - \boldsymbol{Y} > 0$, and $X - Y >> 0$, respectively.

For a square matrix, $M \in R^{n\times n}$, the notation $\sigma(M) = \{z \in R \mid \det(zI\text{-}M) = 0\}$ designates its spectrum, and $\lambda_i(\boldsymbol{M}) \in \sigma(\boldsymbol{M})$, $i = 1,\dots,n$, are its eigenvalues. If $\boldsymbol{M}$ is non-negative, then its spectral radius $\rho(\boldsymbol{M})$, that satisfies $|\lambda_i(\boldsymbol{M})| \leq \rho(\boldsymbol{M})$ for $i = 1,...,n$, is an eigenvalue, i.e. $\rho(\boldsymbol{M}) \in \sigma(\boldsymbol{M})$.

Let matrix $M \in R^{n\times n}$ be non-negative and let $G(\boldsymbol{M})$ stand for the directed graph associated with $\boldsymbol{M}$. Let $1 \leq i, j \leq n$. It is said that: • state i has an *access* to state j, denoted by $i \mapsto j$, if there exists a path from node i to node j in $G(\boldsymbol{M})$; • states i and j *communicate*, denoted by $i \leftrightarrow j$, if $i \mapsto j$ and $j \mapsto i$. Matrix $\boldsymbol{M}$ is called *irreducible* if the directed graph $G(\boldsymbol{M})$ is strongly connected, i.e. any two nodes communicate; otherwise $\boldsymbol{M}$ is called *reducible*. [1].

A non-negative matrix $M \in R^{n\times n}$ has a non-negative right eigenvector $\boldsymbol{v} > 0$, which satisfies $\|\boldsymbol{v}\|_\infty = 1$ and a non-negative left eigenvector $\boldsymbol{w} > 0$, which satisfies $\|\boldsymbol{w}\|_\infty = 1$, both of them corresponding to the eigenvalue $\rho(\boldsymbol{M})$. Depending on the irreducibility / reducibility of matrix $\boldsymbol{M}$, we consider the following cases [1]: • If $\boldsymbol{M}$ is *irreducible*, then (i) $\rho(\boldsymbol{M})$ is a simple eigenvalue, called the Perron-Frobenius eigenvalue; (ii) Both eigenvectors $\boldsymbol{v}$, $\boldsymbol{w}$ are positive, i.e. $v >> 0$, $w >> 0$, and are called the right and left, respectively Perron-Frobenius eigenvectors. • If $\boldsymbol{M}$ is *reducible*, then the positiveness / non-negativeness of the eigenvectors $\boldsymbol{v}$, $\boldsymbol{w}$ is related to the matrix communication classes of $\boldsymbol{M}$ [2].

For a non-negative matrix $M \in R^{n\times n}$, the communication relation, which is an equivalence relation on $G(\boldsymbol{M})$, allows partitioning the set of states $\{1, 2,..., n\}$ into the equivalence classes of $\boldsymbol{M}$ [2]. We say that a class α has an access to a class β if there exist $i \in \alpha$ and $j \in \beta$, so that $i \mapsto j$. A class α is called *final*, if α does not have access to any other class; otherwise α is called *non-final*. A class α is called *basic*, if the condition $\rho(\boldsymbol{M}[\alpha]) = \rho(\boldsymbol{M})$ is met, where $\boldsymbol{M}[\alpha]$ denotes the submatrix of $\boldsymbol{M}$ built with the indices in α; otherwise a class α is called *non-basic*, i.e.. if the condition $\rho(\boldsymbol{M}[\alpha]) < \rho(\boldsymbol{M})$ is met. In this paper, the set of final classes of $\boldsymbol{M}$ is denoted by F , whereas $\not{\mathrm{F}}$ stands for the set of non-final classes. Similarly, the set of basic classes od $\boldsymbol{M}$ is denoted by B , whereas $\not{\mathrm{B}}$ stands for the set of non-basic classes.

1.2. The considered category of economic systems

Consider an economic system with *n* sectors labeled $i = 1,\ldots,n$. The input of sector *i* is denoted by y_i , and the output by x_i . It is assumed that the system operates in an autonomous manner, in the sense that the input of each sector *i* can be ensured by a linear combination of the outputs of the *n* sectors of the economic system, with following form:

$$y_i = \sum_{j=1}^{n} t_{ij} x_j \,,\; t_{ij} \geq 0,\; j = 1,\ldots,n \,,\; i = 1,\ldots,n \,. \tag{1}$$

Equalities of type (1) must be seen as expressed by the help of adequately scaled values, allowing the mathematical connection of the outputs of the various sectors. The autonomous functioning of the economic system is based on the satisfaction of inequalities

$$y_i \leq x_i \,,\; i = 1,\ldots,n \,, \tag{2}$$

which, together with equalities (1), lead to the inequalities

$$\boldsymbol{Tx} \leq \boldsymbol{x} \,,\; T = \left[t_{ij} \right] \in R_+^{n\times n} \,,\; i, j = 1,\ldots,n \,, \tag{3}$$

where the matrix $\boldsymbol{T}$ is non-negative. Inequalities (3) provide for the considered economic system a *Leontief model* of autonomous- or closed-type. Details about the use of this model can be found in Leontief's economic work [3], as well as in works devoted to positive (non-negative) systems, such as [2], [4]-[6].

1.3. Objectives and organization of our work

In this article we study the solvability of the inequality set (3) which involves different approaches depending on the irreducibility / reducibility of the matrix $T \in R_+^{n\times n}$. Our investigation will point out how to use the Perron-Frobenius theory in the above-mentioned situations. The situation of matrix $\boldsymbol{T}$ irreducible and $\rho(\boldsymbol{T}) = 1$ is well-known in literature, when inequalities (3) are solved as equalities, the solution being given by the Perron-Frobenius right eigenvector of the matrix $\boldsymbol{T}$. Instead, the literature referring to the other situations of inequalities (3) is rather scarce. The solvability of inequalities (3) is addressed in Chapter 9 of the monograph [2] that relies on the block triangular form of the matrix $\boldsymbol{T}$, and in our recent paper [6] that is based on the properties of the communication classes of the matrix $\boldsymbol{T}$. In both mentioned works the solvability of inequalities (3) is analyzed in qualitative terms, in the sense of the existence or non-existence of certain types of solutions (called equilibrium solutions or feasible solutions – details are available in the next section).

The present article exploits the point of view recently elaborated by [6]; the new developments proposed by the cited paper aim at constructing solutions in quantitative terms, especially for the solutions called feasible. The remaining sections of the article are organized as follows. Section 2 deals with the qualitative analysis of the solutions to inequalities (3), summarizing our results reported in the previous paper. Section 3 sets out a quantitative approach to solvability in terms of construction methods, the actual implementation of which uses the qualitative results in Section 2. Section 4 illustrates the theoretical developments for a fifth-order Leontief system that allows discussing many details in terms of both qualitative analysis and quantitative construction. Section 5 provides a series of final comments on the significance of the research presented in the paper.

2. Qualitative analysis of solutions

To analyze, from a qualitative point of view (i.e. not necessarily on concrete values), the solutions of the system described by a model of form (3), we consider two types of solutions (associated with two distinct sets of solutions) - as shown by the following definition, which was adapted in accordance with the paper [6].

Definition 1. Consider a system described by model (3).

(a) The vector $x >> 0$ represents a *feasible solution* for the considered system, if inequalities (3) are satisfied. The set

$$\mathrm{X}_f = \left\{ x \in R^n \middle| x >> 0, \|\mathrm{x}\|_\infty = 1, Tx \le x \right\} \tag{4}$$

is called the set of feasible solutions.

(b) The vector $x >> 0$ represents an *equilibrium solution* for the considered system if inequalities (3) are satisfied as equalities. The set

$$\mathrm{X}_e = \left\{ x \in R^n \middle| x >> 0, \|\mathrm{x}\|_\infty = 1, Tx = x \right\} \tag{5}$$

is called the set of equilibrium solutions. ■

Subsections 2.1 and 2.2 composing this section draw attention to the fact that the qualitative analysis of solutions can be addressed in two distinct ways that rely on the properties of matrix ***T***, by exploring either the communication classes of the matrix ***T***, or the Perron-Frobenius eigenstructure of matrix ***T***. This qualitative analysis of the solutions is required whenever matrix ***T*** of model (3) is reducible.

2.1. Use of communication classes

The investigation of communication classes is a fundamental procedure (e.g. [2]) for testing the positiveness of the eigenvectors associated with Perron-Frobenius eigenvalue of a reducible matrix. Proposition 1 below extends this procedure to the qualitative analysis of the solutions to inequalities (3).

Proposition 1. Consider a system described by model (3) and the solution sets $\mathbf{X}_f$ (4), $\mathbf{X}_e$ (5). Denote by $v \in R^n$, $\boldsymbol{v} > 0$, $\|\boldsymbol{v}\|_\infty = 1$, the nonnegative right eigenvector of the matrix $\boldsymbol{T}$ that is associated with the eigenvalue $\rho(\boldsymbol{T})$.

(a) If $\rho(\boldsymbol{T}) > 1$, then $\mathbf{X}_e = \mathbf{X}_f = \varnothing$.

(b) If $\rho(\boldsymbol{T}) = 1$ and

  (b-1) $\boldsymbol{T}$ is irreducible, then $\mathbf{X}_e = \mathbf{X}_f = \{\boldsymbol{v}\}$.

  (b-2) $\boldsymbol{T}$ is reducible and $\mathrm{B} \cap \bar{\mathrm{F}} \neq \varnothing$, then $\mathbf{X}_e = \mathbf{X}_f = \varnothing$.

  (b-3) $\boldsymbol{T}$ is reducible and $\mathrm{B} \subseteq \mathrm{F}$, then $\mathbf{X}_f \neq \varnothing$ with the following two subcases:

    (b-3.1) $\mathbf{X}_e = \{v\}$, when $\mathrm{B} \equiv \mathrm{F}$.

    (b-3.2) $\mathbf{X}_e = \varnothing$, when $\bar{\mathrm{B}} \cap \mathrm{F} \neq \varnothing$.

(c) If $\rho(\boldsymbol{T}) < 1$, then $\mathbf{X}_f \neq \varnothing$ and $\mathbf{X}_e = \varnothing$.

Proof: See our previous paper devoted to Leontief systems [6]. ■

The practical use of Proposition 1 is not always convenient, the difficulties being generated by the identification of the communication classes and their types (because real cases generally mean manipulating matrices of large sizes).

2.2. Use of Perron-Frobenius eigenvectors

The objective of the current subsection is to present a solvability analysis for inequalities (3) which does not use the communication classes of matrix $\boldsymbol{T}$, so that the disadvantage discussed at the end of the previous subsection can be avoided. Proposition 2 below exploits the Perron-Frobenius eigenstructure of the matrix $\boldsymbol{T}$, the numerical calculation of which is much easier than the manipulation of equivalence classes. Proposition 2 is formulated for the situation that is most commonly encountered in practice, namely issue (b) of Proposition 1.

Proposition 2. Consider a system described by model (3) and the solution sets $\mathbf{X}_f$ (4), $\mathbf{X}_e$ (5). Assume that $\rho(\boldsymbol{T}) = 1$ is a simple eigenvalue. Denote by $v = [v_1 \dots v_n]^{\mathrm{t}} \in R^n$, $\boldsymbol{v} > 0$, $\|\boldsymbol{v}\|_\infty = 1$, and $w = [w_1 \dots w_n]^{\mathrm{t}} \in R^n$, $\boldsymbol{w} > 0$, $\|\boldsymbol{w}\|_\infty = 1$, the right and left, respectively, eigenvectors of matrix $\boldsymbol{T}$, associated with the eigenvalue $\rho(\boldsymbol{T})$.

(i) There exists a non-empty set of indices $\mathrm{I} \subseteq \{1,\dots,n\}$ such that for any $i \in \mathrm{I}$ the condition $v_i > 0$, $w_i > 0$ is fulfilled.

(ii) Define the set of indices $\mathrm{J} = \{1,\dots,n\} \setminus \mathrm{I}$.

  (ii-1) If $\mathrm{J} = \varnothing$, then $\mathbf{X}_e = \mathbf{X}_f = \{\boldsymbol{v}\}$.

  (ii-2) If $\mathrm{J} \neq \varnothing$ and $w_j > 0$ for at least an index $j \in \mathrm{J}$, then $\mathbf{X}_e = \mathbf{X}_f = \varnothing$.

(ii-3) If $J \neq \varnothing$ and $w_j = 0$ for all indices $j \in J$, then $\mathbf{X}_f \neq \varnothing$, including the following two subcases:

(ii-3.1) $\mathbf{X}_e = \{\boldsymbol{v}\}$, when $v_j > 0$ for all indices $j \in J$;

(ii-3.2) $\mathbf{X}_e = \varnothing$, when $v_j = 0$ for at least an index $j \in J$.

Proof: See our previous paper devoted to Leontief systems [6]. ■

Proposition 2 shows that the direct examination of the eigenvectors associated with the simple eigenvalue $\rho(\boldsymbol{T}) = 1$ allows us to discuss the solvability of inequalities (3), without having to determine the communication classes of matrix $\boldsymbol{T}$. Besides the fact that the Perron-Frobenius right eigenvector is an equilibrium solution - in the case of system (3) satisfied by equalities, the structure of both left and right eigenvectors provides complete information on the existence / non-existence of feasible solutions - in the case of system (3) satisfied by inequalities. We emphasize the idea that this information is only of a qualitative nature, and that the next section will deal with the construction of solutions corresponding to concrete numerical values.

3. Construction of solutions

The current section is devoted to the proper construction of feasible solutions corresponding to situation (ii-3) of Proposition 2 stated in the previous section. The result below (Proposition 3) will highlight the construction technique based on the concrete calculation of the eigenvectors for a non-negative matrix that majorizes matrix $\boldsymbol{T}$. At the same time, Proposition 3 also discusses the construction of feasible solutions in the case $\rho(\boldsymbol{T}) < 1$ - case which has not been addressed in qualitative terms by Proposition 2 (to ensure the simplicity of exposure).

Proposition 3. Denote by $\Delta \in R^{n \times n}$, $\Delta \geq 0$, a non-negative matrix, by $\boldsymbol{T}_\Delta$ a matrix built as $\boldsymbol{T}_\Delta = \boldsymbol{T} + \boldsymbol{\Delta}$, and by $v_\Delta \in R^n$, $\boldsymbol{v}_\Delta > 0$, $\| \boldsymbol{v}_\Delta \|_\infty = 1$, the right eigenvector of $\boldsymbol{T}_\Delta$ associated with the spectral radius $\rho(\boldsymbol{T}_\Delta)$.

(a) Consider a Leontief system of form (3) and the hypothesis of Proposition 2 (ii-3). There exists a set of non-negative matrices $\mathrm{D_a} \in R_+^{n \times n}$ such that for any $\Delta \in \mathrm{D_a}$ the right eigenvector $\boldsymbol{v}_\Delta$ of matrix $\boldsymbol{T}_\Delta$ represents a feasible solution, i.e. $\boldsymbol{v}_\Delta \in \mathbf{X}_f$.

(b) Consider a Leontief system of form (3) with $\rho(\boldsymbol{T}) < 1$. There exists a set of non-negative matrices $\mathrm{D_b} \in R_+^{n \times n}$ such that for any $\Delta \in \mathrm{D_b}$ the right eigenvector $\boldsymbol{v}_\Delta$ of matrix $\boldsymbol{T}_\Delta$ represents a feasible solution, i.e. $\boldsymbol{v}_\Delta \in \mathbf{X}_f$.

Proof: The proof is constructive for both case (a) and case (b). For the perturbation matrix $\Delta \geq 0$ and for the componentwise majorized matrix $\boldsymbol{T}_\Delta = \boldsymbol{T} + \boldsymbol{\Delta}$ we can write $\tilde{\Delta} = \boldsymbol{P}\Delta\boldsymbol{P}^{\mathrm{t}} \geq 0$ of form (6) and $\tilde{\boldsymbol{T}}_{\Delta} = \boldsymbol{P}\boldsymbol{T}_{\Delta}\boldsymbol{P}^{\mathrm{t}} = \tilde{\boldsymbol{T}} + \tilde{\Delta}$, respectively, where $\boldsymbol{P}$ denotes a permutation matrix.

At the same time, the proof exploits a series of constructive details presented by our earlier work on Leontief systems [6], which are not entirely reproduced by the current text. Relying on these details, one can conclude that the situations in Proposition 1 (b) refer to the following cases:

(I) $\boldsymbol{T}$ irreducible – corresponds to hypothesis (b-1).

(II) $\boldsymbol{T}$ reducible and $\mathrm{B} \subseteq \mathrm{F}\setminus$ - corresponds to hypothesis (b-2).

(III) $\boldsymbol{T}$ reducible and $\mathrm{B} \equiv \mathrm{F}$ - corresponds to hypothesis (b-3.1).

(IV) $\boldsymbol{T}$ reducible and $\mathrm{B} \subset \mathrm{F}$ - corresponds to hypothesis (b-3.2).

(a) Define the set of non-negative matrices $\mathrm{D}_{\mathrm{a}} \in R_{+}^{n\times n}$ such that for any $\Delta \in \mathrm{D}_{\mathrm{a}}$, the matrix $\tilde{\Delta} = \boldsymbol{P}\Delta\boldsymbol{P}^{\mathrm{t}}$ has form (6) with $\Delta_{gs} \geq 0$, $2 \leq s, g \leq k$, $s \leq g-1$, and at least one block satisfies $\Delta_{gs} > 0$. Notice that the diagonal blocks of the matrix $\tilde{\boldsymbol{T}}$ are found in an identical form in matrix $\tilde{\boldsymbol{T}}_{\Delta}$, fact which means that for the spectral radii of these blocks we can write $\rho([\tilde{\boldsymbol{T}}_{\Delta}]_{ss}) = \rho([\tilde{\boldsymbol{T}}]_{ss}) = \rho(\boldsymbol{T}_{ss})$, $1 \leq s \leq k$.

The hypothesis of Proposition 2 (ii-3) corresponds to cases (III) and (IV) mentioned above, which permit the analysis of the connections between the form of the eigenvectors and the structure of the communication classes.

$$\tilde{\Delta} = \boldsymbol{P}\Delta\boldsymbol{P}^{\mathrm{t}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \Delta_{21} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta_{(g-1)1} & \Delta_{(g-1)2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \Delta_{g1} & \Delta_{g2} & \cdots & \Delta_{g(g-1)} & 0 & 0 & \cdots & 0 & 0 \\ \Delta_{(g+1)1} & \Delta_{(g+1)2} & \cdots & \Delta_{(g+1)(g-1)} & \Delta_{(g+1)g} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta_{(k-1)1} & \Delta_{(k-1)2} & \cdots & \Delta_{(k-1)(g-1)} & \Delta_{(k-1)g} & \Delta_{(k-1)(g+1)} & \cdots & 0 & 0 \\ \Delta_{k1} & \Delta_{k1} & \cdots & \Delta_{k(g-1)} & \Delta_{kg} & \Delta_{k(g+1)} & \cdots & \Delta_{k(k-1)} & 0 \end{bmatrix} \quad (6)$$

• If the structure of matrix $\boldsymbol{T}$ belongs to case (III) (corresponding to hypothesis (ii-3.1)), then, for any perturbation matrix of form (6), the structure of matrix $\tilde{\boldsymbol{T}}_{\Delta}$ also belongs to case (III). Thus, we have

$$\tilde{v}_{\Delta} = P v_{\Delta} >> 0, \quad (7)$$

and we can write

$$\tilde{\boldsymbol{T}}\tilde{\boldsymbol{v}}_{\Delta} < \tilde{\boldsymbol{T}}\tilde{\boldsymbol{v}}_{\Delta} + \tilde{\Delta}\tilde{\boldsymbol{v}}_{\Delta} = (\tilde{\boldsymbol{T}} + \tilde{\Delta})\tilde{\boldsymbol{v}}_{\Delta} = \tilde{\boldsymbol{T}}_{\Delta}\tilde{\boldsymbol{v}}_{\Delta} = \tilde{\boldsymbol{v}}_{\Delta} \quad (8)$$

By using the inverse permutation, we get

$$\boldsymbol{P}^{\mathrm{t}}(\tilde{\boldsymbol{T}}\tilde{\boldsymbol{v}}_{\Delta}) < \boldsymbol{P}^{\mathrm{t}}\tilde{\boldsymbol{v}}_{\Delta} \Leftrightarrow (\boldsymbol{P}^{\mathrm{t}}\tilde{\boldsymbol{T}}\boldsymbol{P})(\boldsymbol{P}^{\mathrm{t}}\tilde{\boldsymbol{v}}_{\Delta}) < \boldsymbol{P}^{\mathrm{t}}\tilde{\boldsymbol{v}}_{\Delta} \Leftrightarrow \boldsymbol{P}\boldsymbol{v}_{\Delta} < \boldsymbol{v}_{\Delta} \quad (9)$$

showing hat $\boldsymbol{v}_{\Delta} \in \mathbf{X}_{f}$. In other words, under the hypothesis (ii-3.1) of Proposition 2, besides the equilibrium solution there also exist feasible solutions generated by the right eigenvector associated with the simple eigenvalue $\rho(\boldsymbol{T}_\Delta)=1$ of the matrices of the form $\boldsymbol{T}_\Delta = \boldsymbol{T} + \boldsymbol{\Delta}$.

• If the structure of matrix $\boldsymbol{T}$ belongs to case (IV) (corresponding to hypothesis (ii-3.2)), then one can apply the case denoted as (IV-b) in our paper [6], where $\rho(\boldsymbol{T}_{11})=1$, $\forall s$, $2 \le s \le k$, $\rho(\boldsymbol{T}_{ss})<1$, and $\exists g$, $2 \le g \le k$, such that $\forall s$, $1 \le s \le g-1$, $\boldsymbol{T}_{gs}=0$ (meaning that $\boldsymbol{T}_{11}$ is the only fundamental block, $\boldsymbol{T}_{11}$ is final, $\boldsymbol{T}_{gg}$, $2 \le g \le k$, is final). Subsequently, the components of the right eigenvector satisfy $v^1 >> 0$; $\boldsymbol{v}^s \ge 0$, $2 \le s \le k$; $\boldsymbol{v}^g = 0$, whereas the components of the left eigenvector satisfy $w^1 >> 0$; $\boldsymbol{w}^s = 0$, $2 \le s \le k$. By considering the index *g*, $2 \le g \le k$, we take $\boldsymbol{\Delta}_{gs} > 0$ for all s, $1 \le s \le g-1$. Thus, for all resulting matrices $\tilde{\boldsymbol{T}}_\Delta$, the structure belongs to case (III), and relations (7) – (9) written above preserve their validity for the current case. In other words, under the hypothesis (ii-3.2) of Proposition 2, there exist feasible solutions generated by the right eigenvector associated with the simple eigenvalue $\rho(\boldsymbol{T}_\Delta)=1$ of the matrices of the form $\boldsymbol{T}_\Delta = \boldsymbol{T} + \boldsymbol{\Delta}$.

(b) If $\rho(\boldsymbol{T})<1$ is a simple eigenvalue of matrix $\boldsymbol{T}$, then we can resolve the vector inequality (3) by the help of Proposition 2 and Proposition 3 (a) applied to the vector inequality $\underline{\boldsymbol{T}}\,\underline{\boldsymbol{x}} \le \underline{\boldsymbol{x}}$, where the matrix $\underline{\boldsymbol{T}} = (1/\rho(\boldsymbol{T}))\boldsymbol{T}$ has the simple eigenvalue $\rho(\underline{\boldsymbol{T}})=1$. Indeed, if the vector inequality $\underline{\boldsymbol{T}}\,\underline{\boldsymbol{x}} \le \underline{\boldsymbol{x}}$ is true, then the vector inequality $T\underline{x} << \underline{T}\underline{x} \le \underline{x}$ is also true, that is $\underline{\boldsymbol{x}} \in \mathbf{X}_f$. This approach has a disadvantage, namely that it cannot build solutions when matrix $\boldsymbol{T}$ is reducible and $\mathrm{B} \cap \mathrm{F} \ne \varnothing$, although these solutions exist (according to Proposition 1 (c)). The disadvantage is due to the fact that the reducibility and the structure of the communication classes are essential issues only for the spectral radius equal to 1. Although the original form of the vector inequality (3) considers $\rho(\boldsymbol{T})<1$, the approach presented by us is supposed to resolve $\underline{\boldsymbol{T}}\,\underline{\boldsymbol{x}} \le \underline{\boldsymbol{x}}$, with $\rho(\underline{\boldsymbol{T}})=1$. To eliminate this disadvantage, we can define the set of positive matrices $\mathrm{D}_a \in R_+^{n\times n}$ of form

$$\Delta(\delta) = \delta \mathbf{1}_{n\times n},\ \delta > 0, \tag{10}$$

where $\mathbf{1}_{n\times n}$ denotes the matrix of size $n \times n$ with all entries 1. By using these perturbation matrices, the majorizing matrices $\boldsymbol{T}_{\Delta(\delta)} = \boldsymbol{T} + \Delta(\delta)$ are irreducible, and the right eigenvector associated with $\rho(\boldsymbol{T}_{\Delta(\delta)})$ is positive, that is $v_{\Delta(\delta)} >> 0$. Additionally, the value $\delta > 0$ can be chosen so that $\rho(\boldsymbol{T}_{\Delta(\delta)})$ is as close to $\rho(\boldsymbol{T}_{\Delta(\delta)})$ as we want. Indeed, Lemma 3 in [7] shows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $0 < \delta < \delta(\varepsilon)$ we have $\rho(\boldsymbol{T}) < \rho(\boldsymbol{T}_{\Delta(\delta)}) < \rho(\boldsymbol{T}) + \varepsilon$.

Subsequently, once the condition $\rho(\boldsymbol{T}) < 1$ is fulfilled, we will always be able to build a set of matrices $\Delta(\delta) >> 0$ for which $\rho(\boldsymbol{T}_{\Delta(\delta)}) < 1$ and we can write

$$\begin{aligned} Tv_{\Delta(\delta)} << Tv_{\Delta(\delta)} + \Delta(\delta)v_{\Delta(\delta)} = (T + \Delta(\delta))v_{\Delta(\delta)} \\ = T_{\Delta(\delta)}v_{\Delta(\delta)} = \rho(T_{\Delta(\delta)})v_{\Delta(\delta)} << v_{\Delta(\delta)}. \end{aligned} \tag{11}$$

Thus, the proof of both parts (a) and (b) is completed. ∎

The proof of Proposition 3 shows that the construction procedure for feasible solutions requires the identification of communication classes and their types. The apparent advantage of the strategy operating only with the eigenvectors (without the exploitation of the information provided by the communication classes) is actually limited to the qualitative analysis of the solutions to inequalities (3).

4. Case study

Consider an economic system described by a model of form (3) as follows

$$\begin{aligned} &0.3x_1 + 0.6x_5 \le x_1 \\ &0.5x_2 + 0.5x_4 \le x_2 \\ &0.2x_1 + 0.1x_2 + 0.4x_3 + 0.1x_4 + 0.3x_5 \le x_3 \\ &0.7x_2 + 0.3x_4 \le x_4 \\ &0.6x_1 + 0.2x_5 \le x_5 \end{aligned} \tag{12}$$

4.1. Solutions to system (12) – Qualitative analysis

For the system matrix

$$\boldsymbol{T} = \begin{bmatrix} 0.3 & 0 & 0 & 0 & 0.6 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0.3 \\ 0 & 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0 & 0 & 0 & 0.2 \end{bmatrix} \tag{13}$$

the eigenvalue $\rho(\boldsymbol{T}) = 1$ is simple, and its associated right and left eigenvectors are

$$\boldsymbol{v} = [0 \;\; 1.0000 \;\; 0.3333 \;\; 1.0000 \;\; 0]^{t}, \tag{14}$$

$$\boldsymbol{w} = [0 \;\; 1.0000 \;\; 0 \;\; 0.7143 \;\; 0]^{t}. \tag{15}$$

In order to apply Proposition 1 we investigate the communication classes of the matrix $\boldsymbol{T}$. They are {1,5}, {2,4}, {3}, as resulting from the permuted matrix

$$\boldsymbol{PTP}^{\mathrm{t}} = \begin{array}{c|ccccc} & 2 & 4 & 1 & 5 & 3 \\ \hline 2 & 0.5 & 0.5 & 0 & 0 & 0 \\ 4 & 0.7 & 0.3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0.3 & 0.6 & 0 \\ 5 & 0 & 0 & 0.6 & 0.2 & 0 \\ 3 & 0.1 & 0.1 & 0.2 & 0.3 & 0.4 \end{array}, \quad \boldsymbol{P} = \begin{bmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_4 \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_5 \\ \boldsymbol{e}_3 \end{bmatrix}. \tag{16}$$

The spectral radii of the three blocks are $\rho(\boldsymbol{M}[2,4]) = 1$, $\rho(\boldsymbol{M}[1,5]) = 0.8521$, and $\rho(\boldsymbol{M}[3]) = 0.4$. These classes correspond to the strongly connected subgraphs of the graph $G(\boldsymbol{T})$ as illustrated in fig. 1. Class $\{1,5\}$ is non-basic and final; class $\{2,4\}$ is basic and final; class $\{3\}$ is non-basic and non-final. Subsequently, $\mathrm{B} \subseteq \mathrm{F}$, with $\mathrm{B} \subset \mathrm{F}$ and the statement (b-3.2) of Proposition 1 yields $\mathbf{X}_e = \varnothing$, $\mathbf{X}_f \neq \varnothing$.

We also investigate the structure of the eigenvectors associated with the eigenvalue $\rho(\boldsymbol{T}) = 1$, in order to apply Proposition 2. Hence $\mathrm{J} = \{1,3,5\}$, with $v_1 = 0$, $w_1 = 0$, $v_3 > 0$, $w_3 = 0$, $v_5 = 0$, $w_5 = 0$, and the statement (ii-3.2) of Proposition 2 yields $\mathbf{X}_e = \varnothing$, $\mathbf{X}_f \neq \varnothing$.

As expected, the qualitative analysis of solutions based on Proposition 1 and Proposition 2, respectively, leads to the same result.

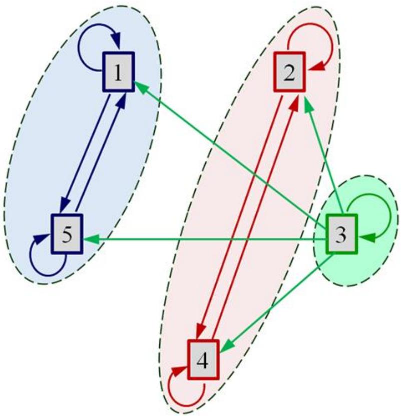

Fig. 1. Strongly connected subgraphs corresponding to the graph of matrix $\boldsymbol{T}$.

4.2. Solutions to system (12) – Construction

By applying Proposition 3(a) we can build matrices $\boldsymbol{T}_\Delta = \boldsymbol{T} + \boldsymbol{\Delta}$ majorizing matrix $\boldsymbol{T}$, which are able to change the type of class $\{1,5\}$ - from non-basic and final, to non-basic and non-final. For example, such a matrix is

$$\boldsymbol{T}_\Delta = \boldsymbol{T} + \Delta = \begin{bmatrix} 0.3 & 0.001 & 0 & 0.001 & 0.6 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0.3 \\ 0 & 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0.001 & 0 & 0.001 & 0.2 \end{bmatrix}, \quad (17)$$

where

$$\Delta = \begin{bmatrix} 0 & 0.001 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0.001 & 0 \end{bmatrix}. \quad (18)$$

The majorizant matrix $\boldsymbol{T}_\Delta$ has the same communication classes as $\boldsymbol{T}$, in accordance with the permuted matrix

$$\boldsymbol{P}\boldsymbol{T}_\Delta \boldsymbol{P}^{t} = \begin{array}{c} \\ 2 \\ 4 \\ 1 \\ 5 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 2 & 4 & 1 & 5 & 3 \end{array} \\ \left[\begin{array}{cc|cc|c} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 & 0 \\ \hline 0.001 & 0.001 & 0.3 & 0.6 & 0 \\ 0.001 & 0.001 & 0.6 & 0.2 & 0 \\ \hline 0.1 & 0.1 & 0.2 & 0.3 & 0.4 \end{array}\right] \end{array}. \quad (19)$$

Instead, for matrix $\boldsymbol{T}_\Delta$, the class {1,5} is non-basic and non-final, and the class {2,4} is the only basic class and the only final class. Subsequently, a feasible solution to inequalities (3) (but not equilibrium solution) is ensured by the positive vector

$$\boldsymbol{v}_\Delta = [0.0140 \;\; 1 \;\; 0.3445 \;\; 1 \;\; 0.0130]^{t} \in \mathbf{X}_f \quad (20)$$

representing the right eigenvector of matrix $\boldsymbol{T}_\Delta$, associated with the eigenvalue $\rho(\boldsymbol{T}_\Delta) = \rho(\boldsymbol{T}) = 1$. A generalization of the above numerical construction is a straightforward task. For any perturbation matrix Δ having the positive entries $\Delta_{1,2} > 0$, $\Delta_{1,4} > 0$, $\Delta_{5,2} > 0$, $\Delta_{1,4} > 0$, and all other entries 0, a majorizant matrix $\boldsymbol{T}_\Delta$ is obtained, the right eigenvector of which is positive and represents a feasible solution of inequalities (3). At the same time, it will be noticed that, regardless of the concrete values of the non-zero entries of the matrix Δ (with the structure considered above), the left eigenvector remains unchanged, that is

$$\boldsymbol{w}_\Delta = \boldsymbol{w} = [0 \;\; 1.0000 \;\; 0 \;\; 0.7143 \;\; 0]^{t}. \quad (21)$$

5. Concluding remarks

The paper creates a deeper insight into the framework of algebraic connections between the non-negative matrices defining Leontief models and the solutions that these models can have. It is shown that the Perron-Frobenius theory is an effective tool for both qualitative and quantitative approach. Qualitative treatment involves the solvability analysis that guarantees the existence or absence of solutions, while quantitative treatment involves construction procedures that provide concrete solutions. Both aspects are equally highlighted by a case study considered to offer practical illustration for the conceptual values presented in the paper.

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