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Improving the behaviour of a suboptimal control system

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Abstract. The discrete-time optimal control law denoted fhan is often used in practical implementations as well as in the research field of control systems. The present paper shows that by reevaluating the design conditions, the control loop driven by fhan achieves a permanent oscillating regime instead of a stable state. Under these circumstances, the fhan control law becomes a suboptimal solution. To further improve the behaviour, a complementary control action is proposed based on the proper discrete-time model of the double-integrator plant in order to eliminate the oscillating regime and stabilize the control loop. The effects of the proposed solution are illustrated through examples. From the practical point of view, implementing the solution is immediate because only the resources needed for the implementation of fhan control law are used.

Keywords. fhan control law, discrete-time optimal control, suboptimality, double-integrator plant, discretization methods of time-continuous linear systems.

1. Introduction

One of the themes developed during the pioneering age of optimal control is the time optimal control of continuous-time processes [1], [2]. The use of digital technologies has led to the adaptation of the problem for optimal control of discrete-time systems. Generally, the methods developed for both categories of systems have led to solutions that can be implemented only by applying numerical methods. For some continuous-time processes, modeled as continuous-time systems, analytical solutions were obtained. Such a case is the double integrator system used as an elementary model for positioning systems [3].
In 2004, in the paper [4] was published a control law with important scientific and practical resonance, often called fhan. Essentially, the great scientific impact

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should be associated with a lot of papers published by Jingqing Han and his collaborators [5], [6].
Our paper discusses the optimality of the solution in [4]. The conclusion is that in fact fhan leads to a suboptimal control but can be improved.
This paper is organized as follows: in Section 2, the time-discrete control law fhan is presented. Discretization methods of time-continuous systems applicable to the time optimal control problem are briefly exposed in Section 3. The optimality of fhan is discussed in Section 4. A solution for improving the behaviour of fhan control law in permanent regime is proposed in Section 5. Finally, some conclusions are drawn in Section 6.

2. The control law fhan

Fhan is the non-linear discrete-time control law (2) that replaces the time optimal control of the continuous-time double-integral plant (1) for a sampling time h. The control law fhan is implemented through the state feedback control loop in Fig. 1a.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t), \quad |u(t)| \le r, \tag{1}$$

$$u[k] = -\begin{cases} r \cdot \mathrm{sign}\,(a[k]), & |a[k]| > r \cdot h \\ \dfrac{a[k]}{h}, & |a[k]| \le r \cdot h \end{cases}, \tag{2}$$

where

$$a[k] = \begin{cases} x_2[k] + \dfrac{\sqrt{(r \cdot h)^2 + 8r\,|x_1[k] + h \cdot x_2[k]|} - r \cdot h}{2} \cdot \mathrm{sign}\,(x_1[k] + h \cdot x_2[k]), \\ \qquad\qquad |x_1[k] + h \cdot x_2[k]| > h^2 \cdot r \\ x_2[k] + \dfrac{x_1[k] + h \cdot x_2[k]}{h}, \quad |x_1[k] + h \cdot x_2[k]| < h^2 \cdot r \end{cases} \tag{3}$$

$u[k] = u(k \cdot h),\ x_1[k] = x_1(k \cdot h),\ \ldots$

According to the paper [4] the control law (2) was designed considering the plant (1) via the linear discrete-time realization

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ h \end{bmatrix} \cdot u[k]\ ,\ |u[k]| \le r\ , \tag{4}$$

considering the control loop from Fig. 1b. In the sequel this control loop is denoted by Sd (in some figures as „sd”).

Note: In [4] the expression of u[k] in (2) is denoted by fst. In [6] and then in the numerous papers citing this article the function with which the values u[k] are calculated is denoted by fhan.

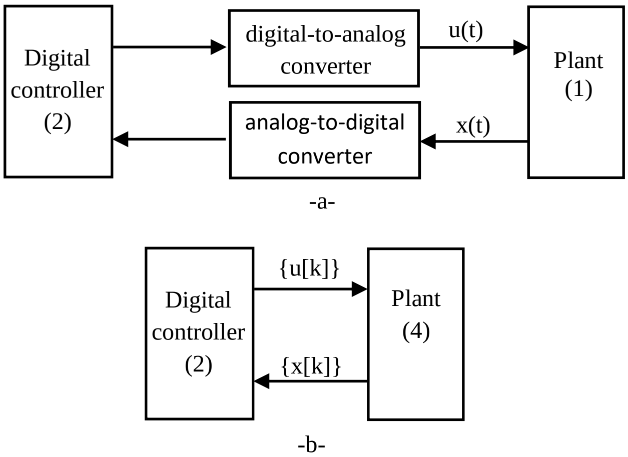

Fig. 1. Regarding the implementation (a) and design (b) of the discrete-time control law fhan

3. Some aspects about the discretization methods of continuous-time linear time-invariant systems

The continuous-time differential equations used as mathematical models for linear time-invariant systems are discretized, by transforming them into discrete difference equations used as mathematical models for discrete-time linear time-invariant systems. Regarding this issue and in respect to discrete-time controllers, articles [7] and [8], published by H. Hanselmann, are of real importance.

Let u(t) be the input signal of a continuous-time linear time-invariant system S and $\{u[k]\}_{k\in N}$, u[k]=u(kh) the corresponding discrete-time signal obtained by using a constant sampling period h. The discretization methods of S depend on the shape of u(t). In the control systems theory are for interest two cases: i) u(t) is a staircase function with steps at the sampling moments; ii) u(t) is a finite values function that does not belong to the staircase function class.

The first case is well known through the following result [9]: the discrete-time system corresponding to the continuous-time linear time-invariant system

$$\begin{cases} \dot{x}(t) = A \cdot x(t) + B \cdot u(t) \\ y(t) = C \cdot x(t) \end{cases} \quad (5)$$

is

$$\begin{cases} x[k+1] = A_d \cdot x[k] + B_d \cdot u[k] \\ y[k] = C \cdot x[k] \end{cases}, \quad (6)$$

where

$$A_d = e^{Ah}, \quad Bd = \int_0^h e^{Av} dv \quad . \quad (7)$$

This result has a very important consequence: If u(t) is a staircase function with steps of length h or a multiple of h, and {u[k]} is the corresponding discrete-time signal obtained by sampling u(t) at instants k·h, then the responses of systems (5) and (6) coincide in every sampling instant. In this context the system (6) was named invariant realization to step signal of system (5) [6].
Applying formulas (7) to system (1), instead of system (4), system (8) is obtained:

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0.5h^2 \\ h \end{bmatrix} \cdot u[k] \ , \ |u[k]| \le r \ .(8)$$

In the second case, because it is impossible to obtain invariant realizations, the discrete-time models are obtained using approximations methods, namely numerical methods for integro-differential equations. The most important methods are based either on the approximation of the derivation operation over time (ii-1) or on the approximation of the integration operation over time (ii-2).
There are a lot of approximation rules that belong to ii-1. From the numerical differentiation methods the most common method consists in using the formula (9) known as Newton's difference quotient or as first-order progressive approximation of the derivative (forward difference). In (9) x is a scalar variable.

$$\dot{x}(t) \cong \frac{x(t+h) - x(t)}{h} \xrightarrow{t=k\cdot h} \dot{x}(t) \cong \frac{x[k+1] - x[k]}{h} \qquad (9)$$

Using the above in (1) formula the discrete-time system (4) is obtained.
The methods ii-2 belong to the so called substitution methods. Under this name are gathered: backward Euler method, Tustin method (or trapezoidal rule) and forward Euler method. All this methods approximate the integral in the second equality from (10) in different manners. The variables in (10) are scalars.

$$\dot{y}(t) = u(t),\ t \in [k\cdot h, (k+1)\cdot h] \quad \to \quad y((k+1)\cdot h) = y(k\cdot h) + \int_{k\cdot h}^{(k+1)\cdot h} u(t)dt\,. \ (10)$$

4. Discussion

In practical implementations of a digital controller the control signal u(t) applied to the input of a continuous-time plant is a staircase function. In simulink this signal corresponds to the output of a zero order holder.
Consequently, in the case of numerical control of the plant (1), the control signal will be a staircase function, and to investigate the behaviour of the resulting discrete-time control system the model (8) of the plant must be used. The corresponding control loop is shown in Fig. 2 and will be further denoted by Sr (in some figures „sr"). Because fhan was designed by taking into account the model (4) for the plant, instead of model (8), fhan cannot be considered as a „discrete-

time optimal control law", but as a „discrete-time suboptimal control solution". Two issues must further be highlighted.

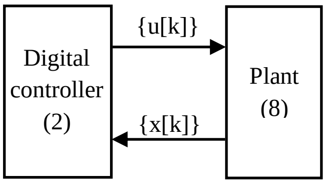

Fig. 2. The control loop Sr.

The suboptimal character explains the success of fhan control law in practical applications. The basic structure discussed in [9] has a wide application potential. Although the structure Sd, composed of plant model (4) and controller (2), is theoretical, its behaviour can be imposed as a reference model for many control loops. Trying to redesign the control law fhan by operating with the plant model (8), we find that this it is no longer possible. The rigorous reasoning of [4], based on isochronic regions, can no longer be repeated. The isochronic regions corresponding to the model (8) are no longer so easy to manipulate as in [4]. Also, from a mathematical point of view there is no longer the possibility of simplifying the amounts that arise as the operating point of the system moves into the state plane.

Naturally, we have to ask what tribute is paid to the suboptimal character of control law. Because fhan is a nonlinear controller a rigorous analysis of the control system Sr, composed of (2) and (8), seems to be rather difficult and is beyond the purpose of this work. Therefore, in this paper we will limit ourselves to observing a single consequence. The chosen scenario is very simple: we compare the behaviour of the system Sd with the behaviour of the system Sr via their free responses (zero-input responses) for different initial conditions. The first control loop, corresponds to the situation for which Fhan was designed. The related signals are drawn in dotted line. The second situation corresponds to fhan's real operating situation. The signals are drawn in continuous line. Both system should be stable. Also, finally they should reach the origin x_1=0, x_2=0.

Figures 3 and 4 illustrate the behaviour of the systems Sd and Sr for different initial conditions ($x_1(0)$ and $x_2(0)$), and different values of r (the maximum absolute value of the control signal), but for the same h (h = 0.1 s).

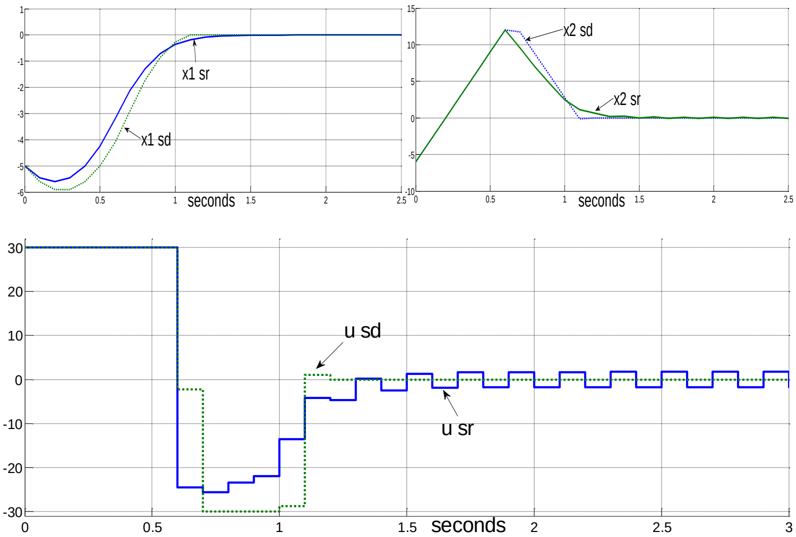

Fig. 3. Case A: $x_1(0)$= -5; $x_2(0)$= -6; r=30; h=0.1 s. (dotted line for sd, continuous line for sr)

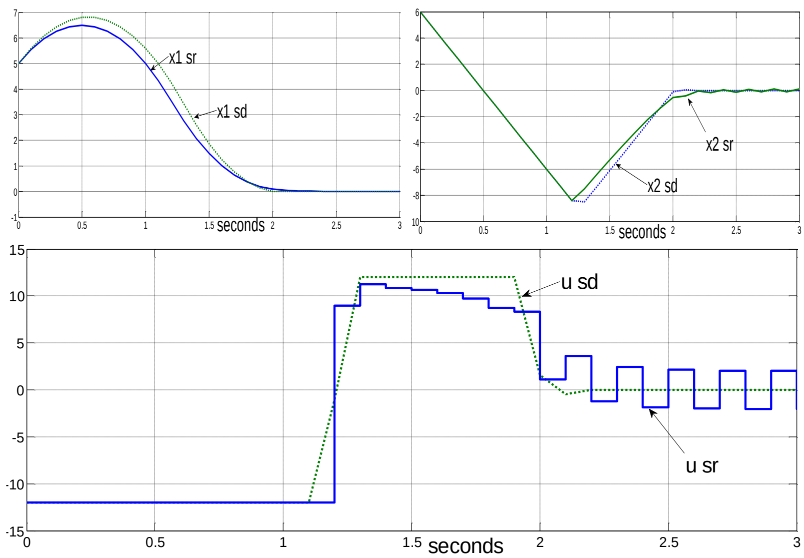

Fig. 4. Case B: $x_1(0)$= 5; $x_2(0)$= 6; r=12; h=0.1 s. (dotted line for sd, continuous line for sr)

In general, the system (1), the double-integrator, is known as the positioning system. The variable x_1 means position (for translational or rotational motion), the variable x_2 means velocity, and the variable u means force or torque. This interpretation simplifies the understanding of the results.

All figures illustrate that both systems, after a transient regime, get into a permanent regime. No significant differences in the transient regime are present. A major difference, however, occurs in the permanent regime. For the system Sd the control u(t) finally stabilizes to u = 0, whereas for the system Sr the control does not stabilize but reaches a permanent oscillating regime with the frequency f = 5 Hz. Naturally, the oscillations appear also in the state variables, but at the chosen scale of representation in the four figures they are not visible.
Intuitively, we realize that the oscillation frequency depends on the value of the sampling period h. Fig. 5 illustrates what happens in a new case C, when h is modified to 0.01 s. The oscillation frequency increases to f=50 Hz.

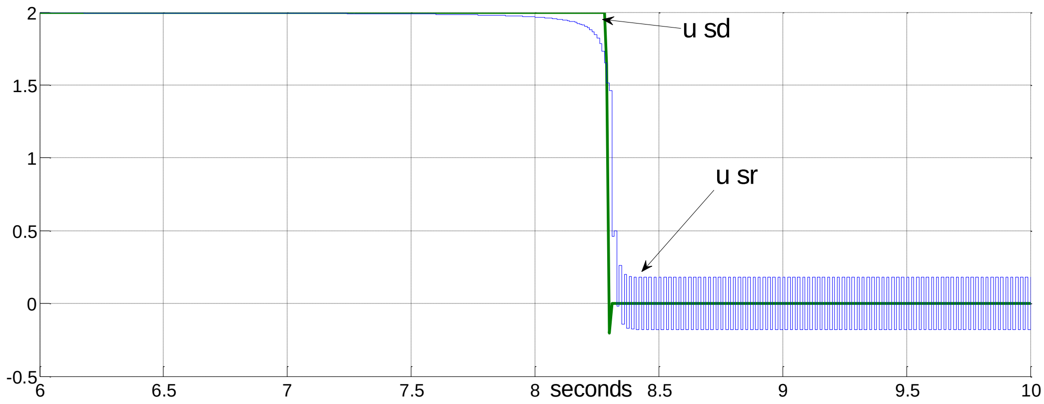

Fig. 5. Case C: $x_1(0)$= 5; $x_2(0)$= 6; r=2; h=0.01 s. (dotted line for sd, continuous line for sr)

If h is further reduced to h=0.001 s the oscillation frequency rises to 500 Hz. Empirically, the equation h·f = 0.5 seems to be valid.

5. Improving the behaviour in permanent regime

The behaviour of control loop Sr (Fig. 2) in permanent regime can be improved based on the model (8). Let us note the values of x_1, x_2 and u for two consecutive instants k and k + 1 with

$$x_1[k] = x_{10}\,,\; x_2[k] = x_{20},\; x_1[k+1] = x_{11}\,,\; x_2[k+1] = x_{21},\; u[k] = u_0 \qquad (11)$$

Then, equations (8) become:

$$\begin{cases} x_{11} = x_{10} + h \cdot x_{20} + 0.5 \cdot h^2 \cdot u_0 \\ x_{21} = x_{20} + h \cdot u_0 \end{cases} ,\quad |u_0| \le r\,. \qquad (12)$$

We call these equations "one-step transition equations". If the transition in one step finishes with the attainment of the origin x_{11}=0 and x_{21}=0, we call this the "last transition".
Now, we focus on designing a two-step control sequence $\{u[t]\} \in \{u[0],\, u[1]\}$ capable of driving the system (8) from an initial arbitrary state, $x_1(0)$, $x_2(0)$, to the origin.

According to equations (12) through an one-step transition generated by u_0 the system (1) reaches the new state $[x_{11}, x_{21}]^T$ starting from the state $[x_{10}, x_{20}]^T$. The new state corresponds to a line segment located on the line (d_0) of equation (14), located between the border points (x_{11-},x_{21-}) and (x_{11+},x_{21+}), where:

$$\begin{cases} x_{11\_} = x_{10} + h \cdot x_{20} - 0.5 \cdot h^2 \cdot r \\ x_{21\_} = x_{20} - h \cdot r \end{cases} \qquad \begin{cases} x_{11_+} = x_{10} + h \cdot x_{20} + 0.5 \cdot h^2 \cdot r \\ x_{21+} = x_{20} + h \cdot r \end{cases} \qquad (13)$$

$$(d_0):\ x_{11} - 0.5 \cdot h \cdot x_{21} - (x_{10} + 0.5 \cdot h \cdot x_{20}) = 0 \qquad (14)$$

If in (14) we set the new state, the new equation will provide the points from which the new state can be accessed. These belong to a segment bordered by points

$$\begin{cases} x_{10\_} = x_{11} - h \cdot x_{21} - 0.5 \cdot h^2 \cdot r \\ x_{20\_} = x_{21} + h \cdot r \end{cases} \qquad \begin{cases} x_{10_+} = x_{11} - h \cdot x_{20} + 0.5 \cdot h^2 \cdot r \\ x_{20+} = x_{21} - h \cdot r \end{cases} \qquad (15)$$

When $[x_{11}, x_{21}]^T$ is the origin, from (14) and (15) results the line segment (16) shown in Fig. 6, bordered by the points P_{1-} and P_{1+}. The line segment [$P_{1-}P_{1+}$] is denoted as domain D[1].

$$(D[1]):\ x_1 + 0.5 \cdot h \cdot x_2 = 0,\ (P_{1-}): \begin{cases} x_{10\_} = -0.5 \cdot h^2 \cdot r \\ x_{20\_} = h \cdot r \end{cases},\ (P_{1+}): \begin{cases} x_{10_+} = 0.5 \cdot h^2 \cdot r \\ x_{20+} = -h \cdot r \end{cases} \qquad (16)$$

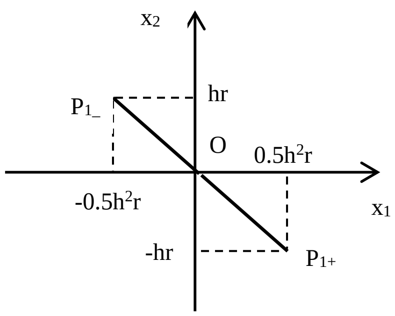

Fig. 6. The domain D[1] represented by the line segment [$P_{1-}P_{1+}$].

From any point $[x_{10}, x_{20}]^T \in [P_{1-}P_{1+}]$ the origin can be reached applying the control:

$$u_0 = \frac{1}{h} x_{20} \quad . \qquad (17)$$

Following the same reasoning for $[x_{11}, x_{21}]^T = [-0.5 \cdot h^2 \cdot r, h \cdot r]^T$ (i.e. for point P_{1-}) and for $[x_{11}, x_{21}]^T = [0.5 \cdot h^2 \cdot r, -h \cdot r]^T$ (i.e. for point P_{1+}), the segments $[x_{10}, x_{20}]^T \in [P_{2--}P_{2-+}]$, and $[x_{10}, x_{20}]^T \in [P_{2+-}P_{2++}]$ respectively are obtained (Fig. 7). The corresponding coordinates are:

$$(P_{2--}):\begin{cases}x_1=-2\cdot h^2\cdot r\\ x_2=2\cdot h\cdot r\end{cases},(P_{2-+}):\begin{cases}x_1=-h^2\cdot r\\ x_2=0\end{cases},(P_{2++}):\begin{cases}x_1=2\cdot h^2\cdot r\\ x_2=-2\cdot h\cdot r\end{cases},(P_{2+-}):\begin{cases}x_1=h^2\cdot r\\ x_2=0\end{cases}$$

(18)

Let D[2] be the domain comprising the parallelogram [$P_{2--}P_{2+-}P_{2++}P_{2-+}$] and its interior Because the points P_{1-} and P_{1+} are located on the sides [$P_{2--}P_{2+-}$] and [$P_{2-+}P_{2++}$] of the parallelogram:

$$D[1]\subset D[2] \quad . \tag{19}$$

D[2] \ D[1] represents the set of states from where D[1] can be reached in one step, and consequently from where the origin can be achieved in two steps. Generally, the state $[x_{11},x_{21}]^T$, corresponding to a point P_x, can be achieved from any point $[x_{10},x_{20}]^T$ belonging to segment [$P_{x-}P_{x+}$] applying the control

$$u_0=-\frac{1}{h^2}x_{10}-\frac{3}{2\cdot h}x_{20} \tag{20}$$

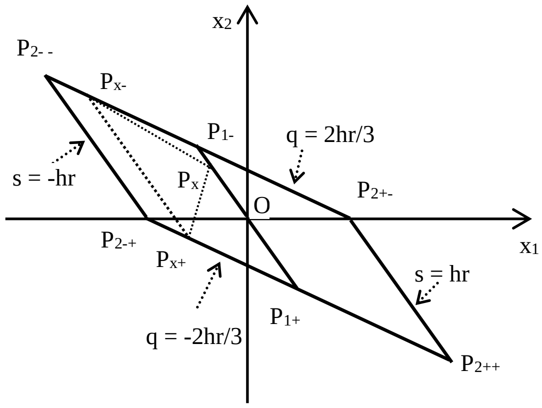

Fig. 7. The domains D[2] and D[1]

The boundaries of D[2] have the equations:

$$s=h\cdot r,\ q=\frac{2}{3}h\cdot r,\ s=-h\cdot r,\ q=-\frac{2}{3}h\cdot r \tag{21}$$

where:

$$s=\frac{2}{h}\cdot x_1+x_2 \quad ,\ q=\frac{2}{3h}\cdot x_1+x_2\,. \tag{22}$$

Note that in terms of s the first equation in (16) becomes s=0.

The above formulas allow the adjustment of the control law (2) based on the rule base (23). The first two rules assure the convergence of the state x(t) to the origin, while the third rule conserves the very good transient behaviour determined by applying the control law fhan:

$$\begin{cases} \text{Rule1: If } [x_1\ x_2]^T \in D[1], \text{ then u is calculated with (17)} \\ \text{Rule2: If } [x_1\ x_2]^T \in D[2], \text{ then u is calculated with (20)} \\ \text{Rule3: } [x_1\ x_2]^T \notin D[1] \text{ and } [x_1\ x_2]^T \notin D[2], \text{ then u is calculated with (2)} \end{cases} \tag{23}$$

For implementation, the schema of Fig. 8 is used. Using a switch, the decision block implements the three rules in (23).

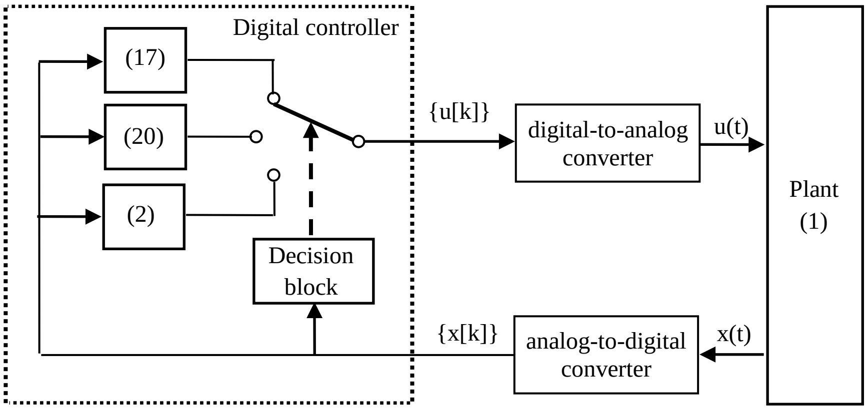

Fig. 8. Implementation of control strategy (23)

In explicit form the strategy (23) is given by (24).

$$\begin{cases} \text{Rule1: If } s[k]=0, |x_2[k]| \le r, \text{ then } u[k] = \frac{1}{h} x_2[k] \\ \text{Rule2: If } |s[k]| \le h \cdot r \text{ and } |q[k]| \le \frac{2}{3} \cdot h \cdot r, \text{ then } u[k] = -\frac{1}{h^2} x_1[k] - \frac{3}{2 \cdot h} x_2[k] \\ \text{Rule3: } \left(s[k]=0, |x_2[k]| > r\right) \text{ and } \left(|s[k]| > h \cdot r \text{ or } |q[k]| > \frac{2}{3} \cdot h \cdot r\right), \\ \qquad \text{then } u[k] = -\begin{cases} r \cdot \mathrm{sign}(a[k]), & |a[k]| > r \cdot h \\ \frac{a[k]}{h}, & |a[k]| \le r \cdot h \end{cases}, \text{ with } a[k] \text{ given by the equation (3)} \end{cases} \tag{24}$$

The results from implementing this strategy are illustrated in Fig. 9 and Fig. 10. The transient regimes obtained applying the control law (24) are denoted with Sr-c.

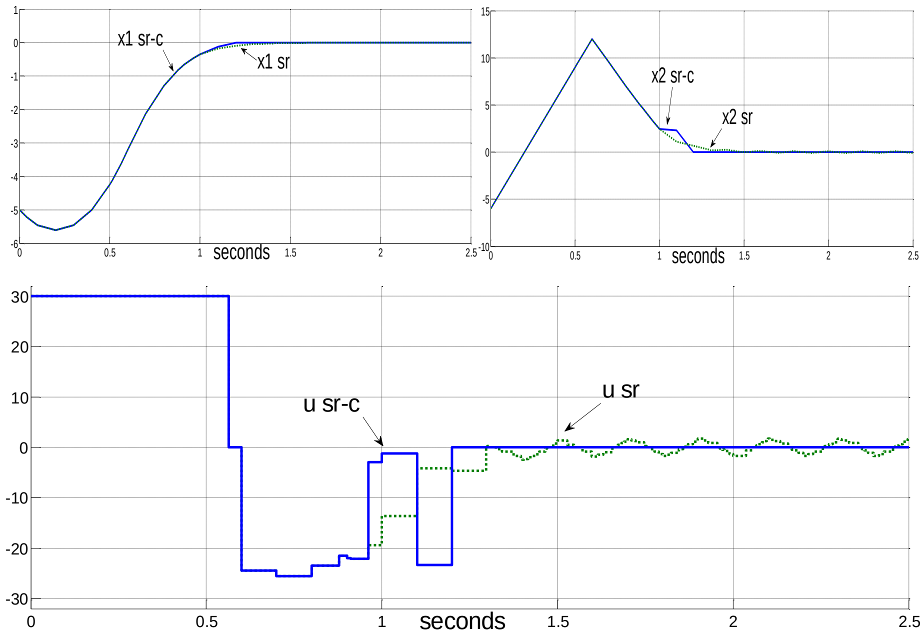

Fig. 9. Case A: $x_1(0)$= -5; $x_2(0)$= -6; r=30; h=0.1 s. (dotted line for sr, continuous line for sr-c)

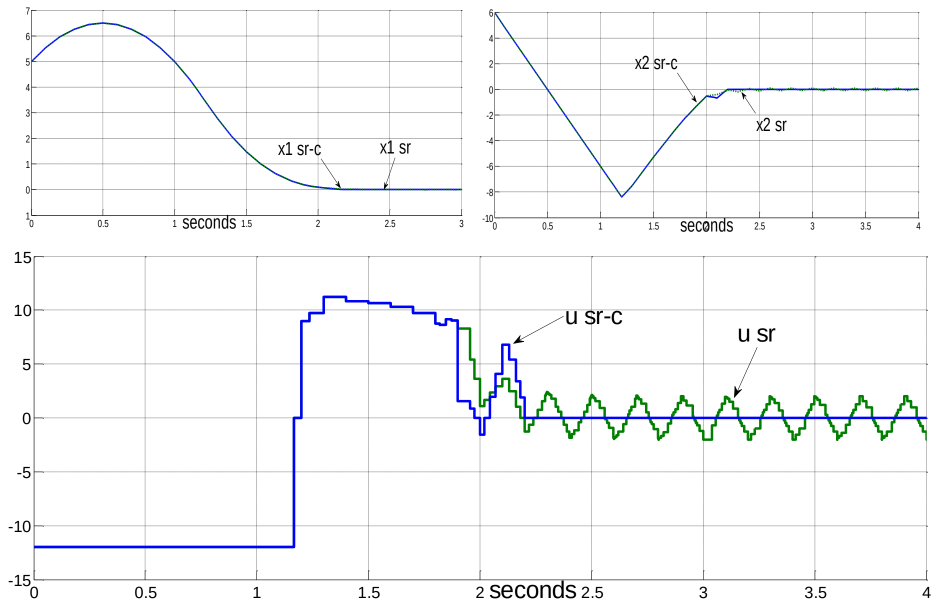

Fig. 10. Case B: $x_1(0)$= 5; $x_2(0)$= 6; r=12; h=0.1 s. (dotted line for sr, continuous line for sr-c)

These results show clearly that fhan (rule 3) brings the system to the neighborhood of origin until the correction actions (rule 1 and rule 2) are activated and the system is then brought to origin. Than, the system is maintained in equilibrium without oscillating of control signal.

6. Conclusions

If the discreet-time nonlinear control law fhan, conceived as an optimal time control for a double-integrator, is verified on the invariant realization to step signal of the double-integrator, it is found that instead of a steady-state regime, i.e. a stable regime, a permanent oscillating regime occurs. This behaviour is due to the design of the fhan control law for a discrete-time model that only approximates the behaviour of the double-integrator plant. Therefore, fhan should be considered only a suboptimal control law.

To improve the behaviour of the control loop by eliminating the permanent oscillating regime, fhan is combined with a two-step control acting around the origin (equilibrium state). The two-step control algorithm is easily obtained using the invariant realization to step signal of the double-integrator. The solution can be implemented with the same effort like fhan.

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