Technical Sciences
Academy of Romania
www.jesi.astr.ro

Journal of Engineering Sciences and Innovation
Volume 2, Issue 4 / 2017, pp. 20-36
http://doi.org/10.56958/jesi.2017.2.4.20

A. Mechanics, Mechanical and Industrial Engineering, Mechatronics

Received ***3 July 2017*** *Accepted* ***4 October 2017***
Received in revised from ***25 September 2017***

On six DOF relative orbital motion of spacecrafts. A complete onboard solution

DANIEL CONDURACHE*

Department of Theoretical Mechanics, Technical University of Iasi,
D. Mangeron Street no.59, 700050, Iasi, Romania

Abstract. In this paper, we reveal a dual-tensor-based procedure to obtain exact expressions for the six degree of freedom (6-DOF) relative orbital motion problem of two spacecrafts, in the specific case of Keplerian confocal orbits. The result is achieved by pure analytical methods in the general case of any leader and deputy motion, without singularities or implying any secular terms. Orthogonal dual tensors play a very important role, with the representation of the solution being, to the authors' knowledge, the shortest approach for describing the complete onboard solution of the 6-DOF orbital motion problem. The solution does not depend on the local-vertical–local-horizontal (LVLH) properties involves that is true in any reference frame of the leader with the origin in its mass centre. A representation theorem is provided for the full-body initial value problem. Furthermore, the representation theorems for rotation part and translation part of the relative motion are obtained.

Keywords: relative orbital motion, full body problem, dual algebra, Lie group, Lie algebra, closed form solution.

1. Introduction

The relative motion between the leader and the deputy in the relative motion is a six-degrees-of-freedom (6-DOF) motion engendered by the joining of the relative translational motion with the rotational one. Recently, the modelling of the 6-DOF motion of spacecraft gained a special attention [1-5], similar to the controlling the relative pose of satellite formation that became a very important research subject [6-10]. The approach implies to consider the relative translational and rotational

*Correspondence address: daniel.condurache@tuiasi.ro

dynamics in the case of chief-deputy spacecraft formation to be modelled using vector and tensor formalism.

In this paper we reveal a dual algebra tensor based procedure to obtain exact expressions for the six D.O.F relative orbital law of motion for the case of two Keplerian confocal orbits.. Orthogonal dual tensors play a very important role, the representation of the solution being, to the authors' knowledge, the shortest approach for describing the complete onboard solution of the six D.O.F relative orbital motion problem. Because the solution does not depend on the LVLH properties involves that is true in any reference frame of the Leader with the origin in its mass centre. To obtain this solution, one has to know only the inertial motion of the Leader spacecraft and the initial conditions of the deputy satellite in the local-vertical-local-horizontal (LVLH) frame. For the full body initial value problem, a general representation theorem is given. More, the real and imaginary parts are split and representation theorems for the rotation and translation parts of the relative orbital motion are obtained. Regarding translation, we will prove that this problem is super-integrable by reducing it to the classic Kepler problem.

The paper is structured as following. The second section is dedicated to the rigid body motion parameterization using orthogonal dual tensors. The Poisson-Darboux problem is extended in dual Lie algebra. In the third section, the state equations for a rigid body motion relative to an arbitrary non-inertial reference frame are determined. Using the obtained result, in the fourth section, the representation theorem and the complete solution for the case of onboard full-body relative orbital motion problem is given. The last section is designated to the conclusions and to the future works.

Nomenclature

a = real number

$\underline{a}$ = dual number

$\mathbf{a}$ = real vector

$\underline{\mathbf{a}}$ = dual vector

$\boldsymbol{A}$ = real tensor

$\underline{\boldsymbol{A}}$ = dual tensor

$\boldsymbol{V}_3$ = real vectors set

$\underline{\boldsymbol{V}}_3$ = dual vectors set

$\boldsymbol{V}_3^{\mathbb{R}}$ = time depending real vectorial functions

$\underline{\boldsymbol{V}}_3^{\mathbb{R}}$ = time depending dual vectorial functions

$\underline{\tilde{a}}$ = skew-symmetric dual tensor corresponding to the dual vector $\underline{\mathbf{a}}$

f_c = true anomaly

p_c = conic parameter

h_c = specific angular momentum of the leader satellite

$L(\underline{V}_3, \underline{V}_3)$ = dual-tensor set

$\mathbb{R}$ = real numbers set

$\underline{\mathbb{R}}$ = dual numbers set

SO_3 = orthogonal real tensors set

$\underline{SO}_3$ = orthogonal dual-tensor set

$SO_3^{\mathbb{R}}$ = time depending real tensorial functions

$\underline{SO}_3^{\mathbb{R}}$ = time depending dual tensorial functions

2. Rigid body motion parameterization using dual Lie algebra

The key notion that will be presented in this section is the tensorial parameterization that can be used to properly describe the rigid-body motion. We discuss the properties of proper orthogonal dual tensorial maps. The proper orthogonal tensorial maps are related with the skew-symmetric tensorial maps via the Poisson-Darboux equation. Orthogonal dual tensorial maps are a powerful instrument in the study of the rigid motion with respect to an inertial and noninertial reference frames. More on dual numbers, dual vectors and dual tensors can be found in the Appendix and in [2]; [16-23].

2.1. Isomorphism between Lie group of the rigid displacements SE_3 and Lie group of the orthogonal dual tensors $\underline{SO}_3$

Let the orthogonal dual tensor set be denoted by:

$$\underline{SO}_3 = \{\underline{R} \in L(\underline{V}_3, \underline{V}_3) | \underline{R}\underline{R}^T = \underline{I}, \det \underline{R} = 1\} \quad (1)$$

where $\underline{SO}_3$ is the set of special orthogonal dual tensors and $\underline{I}$ is the unit orthogonal dual tensor.

The internal structure of any orthogonal dual tensor $\underline{R} \in \underline{SO}_3$ is illustrated in a series of results which were detailed in our previous work [17]; [18]; [23].

Theorem 1. (Structure Theorem). *For any $\underline{R} \in \underline{SO}_3$ a unique decomposition is viable*

$$\underline{R} = (I + \varepsilon \tilde{\rho})Q \quad (2)$$

where $Q \in SO_3$ and $\rho \in V_3$ are called ***structural invariants****,* $\varepsilon^2 = 0$, $\varepsilon \neq 0$.

Taking into account the Lie group structure of $\underline{SO}_3$ and the result presented in previous theorem, it can be concluded that any orthogonal dual tensor $\underline{R} \in \underline{SO}_3$ can be used globally parameterize displacements of rigid bodies.

Theorem 2 (Representation Theorem). *For any orthogonal dual tensor* $\underline{R}$ *defined as in (Eq. (2), a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ *can be computed to have the following equation* [17]; [18]:

$$\underline{R}(\underline{\alpha}, \underline{\mathbf{u}}) = \underline{I} + \sin\underline{\alpha}\,\underline{\tilde{\mathbf{u}}} + (1 - \cos\underline{\alpha})\underline{\tilde{\mathbf{u}}}^2 = \exp(\underline{\alpha}\,\underline{\tilde{\mathbf{u}}}) \tag{3}$$

The parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the **natural invariants** of $\underline{R}$. The unit dual vector $\underline{\mathbf{u}}$ gives the Plücker representation of the Mozzi-Chalses axis [16]; [24] while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translated distance d .

The Lie algebra of the Lie group $\underline{SO}_3$ is the skew-symmetric dual tensor set denoted by $\underline{so}_3 = \{\underline{\tilde{\alpha}} \in \underline{L}(\underline{V}_3, \underline{V}_3) \,|\, \underline{\tilde{\alpha}} = -\underline{\tilde{\alpha}}^T\}$, where the internal mapping is $(\underline{\tilde{\alpha}}_1, \underline{\tilde{\alpha}}_2) = \widetilde{\underline{\alpha}_1 \underline{\alpha}_2}$.

The link between the Lie algebra $\underline{so}_3$, the Lie group $\underline{SO}_3$, and the exponential map is given by the following.

Theorem3. *The mapping*

$$\exp: \underline{so}_3 \to \underline{SO}_3,$$
$$\exp(\underline{\tilde{\alpha}}) = e^{\underline{\tilde{\alpha}}} = \sum_{k=0}^{\infty} \frac{\underline{\tilde{\alpha}}^k}{k!} \tag{4}$$

is well defined and surjective.

Any screw axis that embeds a rigid displacement is parameterized by a unit dual vector, whereas the screw parameters (angle of rotation around the screw and the translation along the screw axis) is structured as a dual angle. The computation of the screw axis is bound to the problem of finding the logarithm of an orthogonal dual tensor $\underline{R}$, that is a multifunction defined by the following equation:

$$\log: \underline{SO}_3 \to \underline{so}_3,$$
$$\log\underline{R} = \{\underline{\tilde{\psi}} \in \underline{so}_3 \,|\, \exp(\underline{\tilde{\psi}}) = \underline{R}\} \tag{5}$$

and is the inverse of (Eq. (4)).

From **Theorem 2** and **Theorem 3**, for any orthogonal dual tensor $\underline{R}$, a dual vector $\underline{\boldsymbol{\psi}} = \underline{\alpha}\,\underline{\mathbf{u}} = \boldsymbol{\psi} + \varepsilon \boldsymbol{\psi}_0$ is computed, represents the **screw dual vector** or **Euler dual vector** (that includes the screw axis and screw parameters) and the form of $\underline{\boldsymbol{\psi}}$ implies that $\underline{\tilde{\psi}} \in \log\underline{R}$. The types of rigid displacements that is parameterized by the Euler dual vector $\underline{\boldsymbol{\psi}}$ as below:

(i) roto-translation if $\boldsymbol{\psi} \neq 0, \boldsymbol{\psi}_0 \neq 0$ and $\boldsymbol{\psi} \cdot \boldsymbol{\psi}_0 \neq 0 \Leftrightarrow |\underline{\boldsymbol{\psi}}| \in \underline{\mathbb{R}}$ and $|\underline{\boldsymbol{\psi}}| \notin \varepsilon\mathbb{R}$;

(ii) pure translation if $\boldsymbol{\psi} = \mathbf{0}$ and $\boldsymbol{\psi}_0 \neq \mathbf{0} \Leftrightarrow \left|\underline{\boldsymbol{\psi}}\right| \in \varepsilon\mathbb{R}$;

(iii) pure rotation if $\boldsymbol{\psi} \neq \mathbf{0}$ and $\boldsymbol{\psi} \cdot \boldsymbol{\psi}_0 = 0 \Leftrightarrow \left|\underline{\boldsymbol{\psi}}\right| \in \mathbb{R}$.

Also, $\|\boldsymbol{\psi}\| < 2\pi$, **Theorem 2** and **Theorem 3** can be used to uniquely recover the screw dual vector $\underline{\boldsymbol{\psi}}$, which is equivalent with computing $\log \underline{\boldsymbol{R}}$.

Theorem 4. *The natural invariants* $\underline{\alpha} = \alpha + \varepsilon d, \underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ *can be used to directly recover the structural invariants* $\boldsymbol{Q}$ *and* $\boldsymbol{\rho}$ *from (Eq. (2))*:

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{I} + \sin\alpha\tilde{\mathbf{u}} + (1-\cos\alpha)\tilde{\mathbf{u}}^2 \\ \boldsymbol{\rho} &= d\mathbf{u} + \sin\alpha\mathbf{u}_0 + (1-\cos\alpha)\mathbf{u} \times \mathbf{u}_0 \end{aligned} \tag{6}$$

To prove (Eq. (6)), we need to use (Eq. (2)) and (Eq. (3)). If these equations are equal, then the structure of their dual parts leads to the result presented in (Eq. (6)).

Theorem 5. (Isomorphism Theorem): *The special Euclidean group* $(\mathbb{SE}_3, \cdot)$ *and* $(\underline{\mathbb{SO}}_3, \cdot)$ *are connected via the isomorphism of the Lie groups*

$$\begin{aligned} &\Phi : \mathbb{SE}_3 \to \underline{\mathbb{SO}}_3, \\ &\Phi(g) = (\boldsymbol{I} + \varepsilon\tilde{\boldsymbol{\rho}})\boldsymbol{Q} \end{aligned} \tag{7}$$

where $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$, $\boldsymbol{Q} \in \mathbb{SO}_3$, $\boldsymbol{\rho} \in \boldsymbol{V}_3$

Proof: For any $g_1, g_2 \in \mathbb{SE}_3$, the map defined in (Eq. (7)) yields

$$\Phi(g_1 \cdot g_2) = \Phi(g_1) \cdot \Phi(g_2) \tag{8}$$

Let $\underline{\boldsymbol{R}} \in \underline{\mathbb{SO}}_3$. Based on **Theorem1**, which ensures a unique decomposition, we can conclude that the only choice for g , such that $\Phi(g) = \underline{\boldsymbol{R}}$ is $g = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}$. This underlines that Φ is a bijection and keeps all the internal operations.

Remark 1: The inverse of Φ is

$$\Phi^{-1} : \underline{\mathbb{SO}}_3 \to \mathbb{SE}_3; \ \Phi^{-1}(\underline{\boldsymbol{R}}) = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix} \tag{9}$$

where $\boldsymbol{Q} = Re(\underline{\boldsymbol{R}}), \boldsymbol{\rho} = vect(Du(\underline{\boldsymbol{R}}) \cdot \boldsymbol{Q}^T)$.

2.2. Poisson-Darboux problems extended in dual Lie algebra

Consider the functions $\boldsymbol{Q} = \boldsymbol{Q}(t) \in \mathbb{SO}_3^{\mathbb{R}}$ and $\boldsymbol{\rho} = \boldsymbol{\rho}(t) \in \boldsymbol{V}_3^{\mathbb{R}}$ to be the parametric equations of any rigid motion. Thus, any rigid motion can be parameterized by a curve in $\underline{\mathbb{SO}}_3$ where $\underline{\boldsymbol{R}}(t) = (\boldsymbol{I} + \varepsilon\tilde{\boldsymbol{\rho}}(t))\boldsymbol{Q}(t)$, where t is time variable. Let $\underline{\mathbf{h}}_0$ embed the Plücker coordinates of a line feature at $t = t_0$. At a time stamp t the line is transformed into:

$$\underline{\mathbf{h}}(t) = \underline{\boldsymbol{R}}(t)\underline{\mathbf{h}}_0 \tag{10}$$

Theorem 6. *In a general rigid motion, described by an orthogonal dual tensor function* $\underline{R}$*, the velocity dual tensor function* $\underline{\Phi}$ *defined as*

$$\dot{\underline{\mathbf{h}}} = \underline{\Phi}\,\underline{\mathbf{h}}, \forall \underline{\mathbf{h}} \in \underline{V}_3 \tag{11}$$

is expressed by

$$\underline{\Phi} = \dot{\underline{R}}\,\underline{R}^{\mathrm{T}}. \tag{12}$$

Let $\underline{\Phi} = \dot{\underline{R}}\,\underline{R}^{\mathrm{T}}$, then $\dot{\underline{R}}\,\underline{R}^{\mathrm{T}} + \underline{R}\,\dot{\underline{R}}^{\mathrm{T}} = \underline{0}$, equivalent with $\underline{\Phi} = -\underline{\Phi}^{\mathrm{T}}$, which shows that $\underline{\Phi} \in \underline{so}_3^{\mathbb{R}}$.

The dual vector $\underline{\omega} = vect\,\dot{\underline{R}}\,\underline{R}^{\mathrm{T}}$ is called dual angular velocity of the rigid body and has the form:

$$\underline{\omega} = \boldsymbol{\omega} + \varepsilon \mathbf{v} \tag{13}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the rigid body and $\mathbf{v} = \dot{\boldsymbol{\rho}} - \boldsymbol{\omega} \times \boldsymbol{\rho}$ represents the linear velocity of the point of the body that coincides instantaneously with the origin of the reference frame. The pair $(\boldsymbol{\omega}, \mathbf{v})$ is usually refereed as the **twist of the rigid body**.

The next Theorem permits the reconstruction of the rigid body motion knowing in any moment the twist of the rigid body that is equivalent with knowing the dual angular velocity [5]; [18].

Theorem 7. *For any continuous function* $\underline{\omega} \in \underline{V}_3^{\mathbb{R}}$ *a unique dual tensor* $\underline{R} \in \underline{SO}_3^{\mathbb{R}}$ *exists so that*

$$\begin{gathered} \dot{\underline{R}} = \underline{\tilde{\omega}}\,\underline{R} \\ \underline{R}(t_0) = \underline{R}_0, \underline{R}_0 \in \underline{SO}_3 \end{gathered} \tag{14}$$

Proof. Consider initially $\underline{R}_0$ to be equal to $\underline{I}$. Equation (14) can be expanded into:

$$\dot{Q} + \varepsilon(\dot{\tilde{\rho}}Q + \tilde{\rho}\dot{Q}) = (\tilde{\omega} + \varepsilon\tilde{v})(Q + \varepsilon\tilde{\rho}Q) = \tilde{\omega}Q + \varepsilon(\tilde{v}Q + \tilde{\omega}\tilde{\rho}Q) \tag{15}$$

the real part of the previous expression leads to:

$$\begin{cases} \dot{Q} = \tilde{\omega}Q \\ Q(t_0) = I \end{cases} \tag{16}$$

Because $\tilde{\omega} = \tilde{\omega}(t)$ is a continuous function,the initial value problem (16) admits a unique solution.

We will prove that this solution is an orthogonal dual tensor.

Denote Q^{T} the transpose of tensor Q . Computing

$$\frac{d}{dt}(QQ^{\mathrm{T}}) = \dot{Q}Q^{\mathrm{T}} + Q\dot{Q}^{\mathrm{T}} = \tilde{\omega}QQ^{\mathrm{T}} - QQ^{\mathrm{T}}\tilde{\omega} = 0 \tag{17}$$

it follows that

$$QQ^{\mathrm{T}} = QQ^{\mathrm{T}}(t_0) = I \tag{18}$$

Since $Q = Q(t)$ is a continuous map, $t \geq t_0$, it follows that $\det(Q)$ is a continuous map too. From (18) it results that $\det(Q) \in \{-1, 1\}$. Since $\det(Q(t_0)) = \det I = 1$, it follows that:

$$\begin{cases} QQ^T = I \\ \det(Q) = 1 \end{cases} \tag{19}$$

Therefore, $Q \in SO_3^{\mathbb{R}}$ is a proper orthogonal tensor map.

The dual part of (15) gives

$$\dot{\tilde{\rho}} + \tilde{\rho}\tilde{\omega} = \tilde{v} + \tilde{\omega}\tilde{\rho} \tag{20}$$

which, taking a step further implies that

$$\dot{\tilde{\rho}} + \tilde{\rho}\tilde{\omega} - \tilde{\omega}\tilde{\rho} = \tilde{v} \tag{21}$$

Using $\widetilde{\omega\rho} = \tilde{\omega}\tilde{\rho} - \tilde{\rho}\tilde{\omega}$ the previous relation is transformed into the differential equation:

$$\begin{cases} \dot{\rho} - \omega \times \rho = v \\ \rho(t_0) = 0 \end{cases} \tag{22}$$

that has the solution

$$\rho = \int_{t_0}^{t} Q(t)Q^T(x)v(x)dx \tag{23}$$

where Q is the solution of (16).

The solution of

$$\begin{cases} \dot{\underline{R}} = \underline{\tilde{\omega}}\,\underline{R} \\ \underline{R}(t_0) = \underline{R}_0, \underline{R}_0 \in \underline{SO}_3 \end{cases} \tag{24}$$

is

$$\underline{R}(t) = \underline{R}_*(t)\underline{R}_0 \tag{25}$$

where $\underline{R}_*(t)$ is the solution of (14) for $\underline{R}_0 = \underline{I}$.

Due to the fact that orthogonal dual tensor $\underline{R}$ completely models the six degree of freedom motion, we can conclude that the **Theorem 7** is the dual form of the **Poisson-Darboux problem** [28] for the case when the rotation tensor is computed from the instantaneous angular velocity. So, in order to recover $\underline{R}$, it is necessary to find out how the dual angular velocity vector $\underline{\omega}$ behaves in time and also the value of $\underline{R}$ at time $t = t_0$.

The dual tensor $\underline{R}$ can be derived from $\underline{\omega}$, when is positioned in space, or from $\underline{\omega}^B$, which denotes the dual angular velocity vector to be positioned in the rigid body.

Remark 2. The dual angular velocity vector positioned in the rigid body can be recovered from $\underline{\omega}^B = \underline{R}^T\underline{\omega}$, thus transforming (Eq. (14)) into:

$$\begin{cases} \dot{\underline{R}} = \underline{R}\,\underline{\tilde{\omega}}^B \\ \underline{R}(t_0) = \underline{R}_0, \underline{R}_0 \in \underline{SO}_3 \end{cases} \tag{26}$$

(Eq. (14)) and (Eq. (26)) represent the dual replica of the classical orientation Poisson-Darboux problem [17]; [28, 29].

The tensorial (Eq. (14)) and (Eq. (26)) are equivalent with 18 scalar differential equations. The dual vectors and dual quaternions parameterizations of the orthogonal dual tensors allow us to determine some solutions of smaller dimension in order to solve the dual Poisson- Darboux problem [5].

3. Rigid body motion in arbitrary non-inertial frame revised

To the author's knowledge, in the field of astrodynamics there aren't many reports on how the motion of rigid body can be studied in arbitrary non-inertial frames. Next, we proposed a dual tensors based model for the motion of the rigid body in arbitrary non-inertial frame. The proposed method eludes the calculus of inertia forces that contributes to the rigid body relative state. So, the free of coordinate state equation of the rigid body motion in arbitrary non-inertial frame will be obtained.

Let $\underline{R}_D$ and $\underline{R}_C$ be the dual orthogonal tensors which describe the motion of two rigid bodies relative to the inertial frame.

If $\underline{R}$ is the orthogonal dual tensor which embeds the six degree of freedom relative motion of rigid body C relative to rigid body D, then:

$$\underline{R} = \underline{R}_C^T \underline{R}_D \tag{27}$$

Let $\underline{\omega}_C$ denote the dual angular velocity of the rigid body C and $\underline{\omega}_D$ the dual angular velocity of the rigid body D, both being related to inertial reference frame. In the followings, the inertial motion of the rigid body C is considered to be known. If $\underline{\omega}$ is the dual angular velocity of the rigid body D relative to the rigid body C, then, conforming with (Eq. (27)):

$$\underline{\omega} = \underline{\omega}_D - \underline{\omega}_C \tag{28}$$

Considering $\underline{\omega}_D^B$ being the dual angular velocity vector of the rigid body D in the body frame, the dual form of the Euler equation given in [30] results that:

$$\underline{M}\,\dot{\underline{\omega}}_D^B + \underline{\omega}_D^B \times \underline{M}\,\underline{\omega}_D^B = \underline{\tau}^B \tag{29}$$

In (Eq. (29)) $\underline{\tau}^B = \mathbf{F}^B + \varepsilon\boldsymbol{\tau}^B$, where $\mathbf{F}^B$ the force applied in the mass centre and $\boldsymbol{\tau}^B$ is the torque. Also in (Eq. (29)), $\underline{M}$ represents the inertia dual operator, which is given by $\underline{M} = m_D \dfrac{d}{d\varepsilon} I + \varepsilon J$, where J is the inertia tensor of the rigid body D related to it's mass centre and m_D is the mass of the rigid body D. Combining $\underline{M}^{-1} = J^{-1}\dfrac{d}{d\varepsilon} + \varepsilon\dfrac{1}{m_D} I$ with (Eq. (29)) results:

$$\dot{\underline{\omega}}_D^B + \underline{M}^{-1}\left(\underline{\omega}_D^B \times \underline{M}\,\underline{\omega}_D^B\right) = \underline{M}^{-1}\underline{\tau}^B \tag{30}$$

Taking into account that $\underline{\omega}_D = \underline{R}\underline{\omega}_D^B$, the dual angular velocity vector can be computed from

$$\underline{\omega} = \underline{R}\,\underline{\omega}_D^B - \underline{\omega}_C \tag{31}$$

this through differentiation gives:

$$\dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \dot{\underline{R}}\,\underline{\omega}_D^B + \underline{R}\,\dot{\underline{\omega}}_D^B \tag{32}$$

If the previous equation is multiplied by $\underline{R}^T$, then

$$\underline{R}^T\left(\dot{\underline{\omega}} + \dot{\underline{\omega}}_C\right) = \underline{R}^T\dot{\underline{R}}\,\underline{\omega}_D^B + \dot{\underline{\omega}}_D^B \tag{33}$$

which combined with $\dot{\underline{R}} = \tilde{\underline{\omega}}\,\underline{R}$ generates:

$$\underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right) = \underline{R}^T \underline{\omega}\, \underline{R}\, \underline{\omega}_D^B + \dot{\underline{\omega}}_D^B \tag{34}$$

After a few steps, (Eq. (34)) is transformed into

$$\dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \underline{R}\, \dot{\underline{\omega}}_D^B + \underline{\omega} \times \underline{\omega}_C \tag{35}$$

which combined with (Eq. (30)) gives:

$$\dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \underline{R}\, \underline{M}^{-1} \underline{\tau}^B - \underline{R}\, \underline{M}^{-1}\left(\underline{\omega}_D^B \times \underline{M}\, \underline{\omega}_D^B\right) + \underline{\omega} \times \underline{\omega}_C \tag{36}$$

Because $\underline{\omega}_D^B = \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right)$, the final equation is:

$$\dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \underline{R}\, \underline{M}^{-1}\left[\underline{\tau}^B - \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right) \times \underline{M}\, \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right)\right] + \underline{\omega} \times \underline{\omega}_C \tag{37}$$

The system:

$$\begin{cases} \dot{\underline{R}} = \tilde{\underline{\omega}}\,\underline{R} \\ \dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \underline{R}\, \underline{M}^{-1}\left[\underline{R}^T \underline{\tau} - \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right) \times @ \times \underline{M}\, \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right)\right] + \underline{\omega} \times \underline{\omega}_C \\ \underline{\omega}(t_0) = \underline{\omega}_0, \underline{\omega}_0 \in \underline{V}_3 \\ \underline{R}(t_0) = \underline{R}_0, \underline{R}_0 \in \underline{SO}_3 \end{cases} \tag{38}$$

is a compact form which can be used to model the six D.O.F relative motion problem. In the previous equation the state of the rigid body D in relation with the rigid body C is modelled by the dual tensor $\underline{R}$ and the dual angular velocities field $\underline{\omega}$. This initial value problem can be used to study the behavior of the rigid body D in relation with the frame attached to the rigid body C. In (Eq. (38)), all the vectors are represented in the body frame of C, which shows that the proposed solution is onboard and has the property of being coupled in $\underline{R}$ and $\underline{\omega}$.

Next, we present a procedure that allows the decoupling of the proposed solution.

In order to describe the solution to (Eq. (38)), we consider the following change of variable:

$$\underline{\omega}_* = \underline{R}^T\left(\underline{\omega} + \underline{\omega}_C\right) \tag{39}$$

This change of variable leads to $\dot{\underline{\omega}}_* = \dot{\underline{R}}^T\left(\underline{\omega} + \underline{\omega}_C\right) + \underline{R}^T\left(\dot{\underline{\omega}} + \dot{\underline{\omega}}_C\right) = -\underline{R}^T \tilde{\underline{\omega}}\left(\underline{\omega} + \underline{\omega}_C\right) + \underline{R}^T\left(\dot{\underline{\omega}} + \dot{\underline{\omega}}_C\right)$. The result is equivalent with $\dot{\underline{\omega}}_* = \underline{R}^T\left(\underline{\omega}_C \times \underline{\omega} + \dot{\underline{\omega}} + \dot{\underline{\omega}}_C\right)$ or

$$\underline{\omega}_C \times \underline{\omega} + \dot{\underline{\omega}} + \dot{\underline{\omega}}_C = \underline{R}\, \dot{\underline{\omega}}_* \tag{40}$$

After some steps of algebraic calculus, from (Eq. (39)), (Eq. (40)) and (Eq. (37)), results that:

$$\begin{cases} \underline{M}\, \dot{\underline{\omega}}_* + \underline{\omega}_* \times \underline{M}\, \underline{\omega}_* = \underline{\tau}_* \\ \underline{\omega}_*(t_0) = \underline{\omega}_*^0 \end{cases} \tag{41}$$

Where $\underline{\tau}_* = \underline{R}^T \underline{\tau}$ is the dual torque related to the mass center in the body frame of the rigid body D and $\underline{\omega}_*^0 = \underline{R}_0^T\left(\underline{\omega}_0 + \underline{\omega}_C(t_0)\right)$. (Eq. (41)) is a dual Euler fixed point classic problem.

For any $\underline{R} \in \underline{SO}_3$, the solution of (Eq. (38)) emerges from

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\tilde{\boldsymbol{\omega}}}\,\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{42}$$

Making use of (Eq. (39)), results that $\underline{\boldsymbol{R}}\,\underline{\boldsymbol{\omega}}_{*} = \underline{\boldsymbol{\omega}} + \underline{\boldsymbol{\omega}}_C$. If $\sim$ operator used, the previous calculus is transformed into $\widetilde{\underline{\boldsymbol{R}}\,\underline{\boldsymbol{\omega}}_{*}} = \underline{\tilde{\boldsymbol{\omega}}} + \underline{\tilde{\boldsymbol{\omega}}}_C \Leftrightarrow \underline{\boldsymbol{R}}\,\underline{\tilde{\boldsymbol{\omega}}}_{*}\underline{\boldsymbol{R}}^T = \dot{\underline{\boldsymbol{R}}}\,\underline{\boldsymbol{R}}^T + \underline{\tilde{\boldsymbol{\omega}}}_C$. After multiplying the last expression by $\underline{\boldsymbol{R}}$, we obtain the initial value problem:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} = \underline{\boldsymbol{R}}\,\underline{\tilde{\boldsymbol{\omega}}}_{*} - \underline{\tilde{\boldsymbol{\omega}}}_C\,\underline{\boldsymbol{R}} \\ \underline{\boldsymbol{R}}(t_0) = \underline{\boldsymbol{R}}_0 \end{cases} \tag{43}$$

Using the variable change (Eq. (39)), the initial value problem (38)) has been decoupled into two distinct initial value problems (41) and (43).

Let $\underline{\boldsymbol{R}}_{-\underline{\omega}_C} \in \underline{SO}_3^{\mathbb{R}}$ be the unique solution of the following Poisson-Darboux problem:

$$\begin{cases} \dot{\underline{\boldsymbol{R}}} + \underline{\tilde{\boldsymbol{\omega}}}_C\,\underline{\boldsymbol{R}} = 0 \\ \underline{\boldsymbol{R}}(t_0) = I - \varepsilon\tilde{\boldsymbol{r}}_C(t_0) \end{cases} \tag{44}$$

Considering $\underline{\boldsymbol{R}} = \underline{\boldsymbol{R}}_{-\underline{\omega}_C}\underline{\boldsymbol{R}}_{*}$, a representation theorem of the solution of (Eq. (38)) can be formulated.

Theorem 8. (Representation Theorem). *The solution of (Eq.* (38)*) results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *from (Eq.* (44)*) to the solution of the classical dual Euler fixed point problem:*

$$\begin{cases} \dot{\underline{\boldsymbol{R}}}_{*} = \underline{\boldsymbol{R}}_{*}\underline{\tilde{\boldsymbol{\omega}}}_{*} \\ \underline{\boldsymbol{M}}\,\dot{\underline{\boldsymbol{\omega}}}_{*} + \underline{\boldsymbol{\omega}}_{*} \times \underline{\boldsymbol{M}}\,\underline{\boldsymbol{\omega}}_{*} = \underline{\boldsymbol{\tau}}_{*} \\ \underline{\boldsymbol{\omega}}_{*}(t_0) = \underline{\boldsymbol{\omega}}_{*0} \\ \underline{\boldsymbol{R}}_{*}(t_0) = \underline{\boldsymbol{R}}_{*0} \end{cases} \tag{45}$$

where $\underline{\boldsymbol{\omega}}_{*0} = \underline{\boldsymbol{R}}_0^T(\underline{\boldsymbol{\omega}}_0 + \underline{\boldsymbol{\omega}}_C(t_0))$, $\underline{\boldsymbol{R}}_{*0} = (I + \varepsilon\tilde{\boldsymbol{r}}_C(t_0))\underline{\boldsymbol{R}}_0$, $\underline{\boldsymbol{\tau}}_{*} = \underline{\boldsymbol{R}}^T\underline{\boldsymbol{\tau}}$.

4. A dual tensor formulation of the six degree of freedom relative orbital motion problem

The results from the previous paragraphs will be used to study the six degrees of freedom relative orbital motion problem.

The relative orbital motion problem may now be considered classical one considering the many scientific papers written on this subject in the last decades. Also, the problem is quite important knowing its numerous applications: rendezvous operations, spacecraft formation flying, distributed spacecraft missions [3, 4]; [6-10].

The model of the relative orbital motion consists in two spacecraft flying in Keplerian orbits due to the influence of the same gravitational attraction centre. The main problem is to determine the pose of the Deputy satellite relative to a reference frame originated in the Leader satellite centre of mass. This non-inertial reference frame, known as "LVLH (Local-Vertical-Local- Horizontal)" is chosen as following: the C_x axis has the same orientation as the position vector of the Leader with respect to an inertial reference frame with the origin in the attraction centre; the orientation of the C_z is the same as the Leader orbit angular momentum; the C_y axis completes a right-handed frame. The angular velocity of the LVLH is given by vector ω_C, which has the expression:

$$\omega_C = \dot{f}_C \frac{\mathbf{h}_C}{\mathbf{h}_C} = \frac{1}{\mathbf{r}_C^2}\mathbf{h}_C = \left[\frac{1 + \mathbf{e}_C \cos f_C(\mathbf{t})}{\mathbf{p}_C}\right]^2 \mathbf{h}_C \quad (46)$$

where vector $\mathbf{r}_C$ is

$$\mathbf{r}_C = \frac{p_C}{1 + e_C \cos f_C(t)} \frac{\mathbf{r}_C^0}{\mathbf{r}_C^0} \quad (47)$$

where p_C is the conic parameter, $\mathbf{h}_C$ is the angular momentum of the Leader, $f_C(t)$ being the true anomaly and e_C is the eccentricity of the Leader.

We propose dual tensors based model for the motion and the pose for the mass centre of the Deputy in relation with LVLH. Both, the Leader satellite and the Deputy satellite can be considered rigid bodies.

Furthermore, the time variation of $\mathbf{r}_C$ is:

$$\dot{\mathbf{r}}_C = \frac{\mathbf{e}_C|\mathbf{h}_C|\sin f_C(\mathbf{t})}{\mathbf{p}_C} \frac{\mathbf{r}_C^0}{\mathbf{r}_C^0} \quad (48)$$

In order to a more easy to read list of notations, for $t = t_0$ there will be used the followings:

$$\omega_C^0 = \left[\frac{1 + \mathbf{e}_C \cos f_C(\mathbf{t}_0)}{\mathbf{p}_C}\right]^2 \mathbf{h}_C \quad (49)$$

$$\mathbf{r}_C^{0\cdot} = \frac{\mathbf{e}_C|\mathbf{h}_C|\sin f_C(\mathbf{t}_0)}{\mathbf{p}_C} \frac{\mathbf{r}_C^0}{\mathbf{r}_C^0} \quad (50)$$

where $\frac{\mathbf{r}_C^0}{\mathbf{r}_C^0}$ is the unity vector of the X-axis from LVLH.

The full-body relative orbital motion is described by the (Eq.(38)) where the dual angular velocity of the Chief satellite is:

$$\underline{\omega}_C = \omega_C + \varepsilon(\dot{\mathbf{r}}_C + \omega_C \times \mathbf{r}_C) \quad (51)$$

and the dual torque related to the mass center of Deputy satellite is:

$$\underline{\tau} = -\frac{\mu}{|\mathbf{r}_c + \mathbf{r}|^3}(\mathbf{r}_c + \mathbf{r}) + \varepsilon\tau\,. \quad (52)$$

The representation theorem (**Theorem8**) is applied in this case using the conditions (48))-(51)), the solution of the Poisson-Darboux problem (44)) is:

$$\underline{R}_{-\underline{\omega}_C} = \left(I - \varepsilon\tilde{r}_C(t)\right)\left(I - \sin f_C^0 \frac{\tilde{h}_C}{\mathbf{h}_c} + \left(1 - \cos f_C^0\right)\frac{\tilde{h}_C^{\,2}}{\mathbf{h}_c^{\,2}}\right). \quad (53)$$

In (53), we've noted $\mathrm{h}_0 = \|\mathbf{h}_0\|$ and $f_c^0 = f_c(t) - f_c(t_0)$.

Theorem 9. (Representation Theorem of the full body relative orbital motion). *The solution of (Eq.* (38)*) results from the application of the tensor* $\underline{\boldsymbol{R}}_{-\underline{\omega}_C}$ *from (Eq.* (53)*) to the solution of the classical dual Euler fixed point problem* (45)*), with* $\underline{\boldsymbol{\omega}}_C$ *and* $\underline{\boldsymbol{\tau}}$ *given by (51) and (52).*

4.1. The rotational and translational parts of the relative orbital motion

The complete solution of (Eq. (38)) can be recovered in two steps.

Consider first the real part of (Eq. (38)). This leads to an initial value problem:

$$\begin{cases} \dot{Q} = \tilde{\omega} Q \\ \dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}}_c = Q J^{-1}\left[Q^T \boldsymbol{\tau} - Q^T(\boldsymbol{\omega} + \boldsymbol{\omega}_c) \times @ \times J Q^T(\boldsymbol{\omega} + \boldsymbol{\omega}_c)\right] + \boldsymbol{\omega} \times \boldsymbol{\omega}_c \\ \boldsymbol{\omega}(\mathrm{t}_0) = \boldsymbol{\omega}_0, \boldsymbol{\omega}_0 \in V_3 \\ Q(\mathrm{t}_0) = Q_0, Q_0 \in SO_3 \end{cases} \tag{54}$$

which has the solution $Q = Q(\mathrm{t})$, the real tensor Q being the attitude of Deputy in relation with LVLH. In (Eq. (54)), $\boldsymbol{\omega}$ is the angular velocity of the Deputy in relation with LVLH, $\boldsymbol{\omega}_c$ is the angular velocity of LVLH, $\boldsymbol{\tau}$ is the resulting torque of the forces applied on the Deputy in relation with is mass centre, J is the inertia tensor of the Deputy in relation with its mass center. The angular velocity of Deputy in respect to LVLH at time t_0 is denoted with $\boldsymbol{\omega}_0$ and Q_0 is the orientation of Deputy in respect to LVLH at time t_0.

Consider now the dual part of (Eq. (38)). Taking into account the internal structure of $\underline{\boldsymbol{R}}$, which is given by (Eq. (2), after some basic algebraic calculus we obtain a second initial value problem that models the translation of the Deputy satellite mass centre with respect to the LVLH reference frame:

$$\begin{cases} \ddot{\mathbf{r}} + 2\boldsymbol{\omega}_c \times \dot{\mathbf{r}} + \boldsymbol{\omega}_c \times (\boldsymbol{\omega}_c \times \mathbf{r}) + \dot{\boldsymbol{\omega}}_c \times \mathbf{r} + \\ + \dfrac{\mu}{\|\mathbf{r}_c + \mathbf{r}\|^3}(\mathbf{r}_c + \mathbf{r}) - \dfrac{\mu}{r_c^3}\mathbf{r}_c = \mathbf{0} \\ \mathbf{r}(t_0) = \mathbf{r}_0, \dot{\mathbf{r}}(t_0) = \mathbf{v}_0 \end{cases} \tag{55}$$

where $\mu > 0$ is the gravitational parameter of the attraction centre and $\mathbf{r}_0, \mathbf{v}_0$ represent the relative position and relative velocity vectors of the mass centre of the Deputy spacecraft with respect to LVLH at the initial moment of time $\mathrm{t}_0 \geq 0$.

Based on the **representation theorem 9,** the following theorem results.

Theorem 10. *The solutions of problems (Eq. (54)) and (Eq. (55)) are given by*

$$\begin{aligned} Q &= R_{-\omega_C} Q_* \\ \mathbf{r} &= R_{-\omega_C} \mathbf{r}_* - \mathbf{r}_c \end{aligned} \tag{56}$$

where Q_* *and* $\mathbf{r}_*$ *are the solutions of the the classical Euler fixed point problem and, respectively, Kepler's problem*:

$$\begin{cases} \dot{Q}_* = Q_* \tilde{\omega}_* \\ J\dot{\omega}_* + \omega_* \times J\omega_* = \tau_* \\ \omega_*(t_0) = Q_0^T(\omega_0 + \omega_c(t_0)) \\ Q_*(t_0) = Q_0 \end{cases} \tag{57}$$

and

$$\begin{cases} \ddot{\mathbf{r}}_* + \dfrac{\mu}{r_*^3}\mathbf{r}_* = \mathbf{0}; \\ \mathbf{r}_*(t_0) = \mathbf{r}_c^0 + \mathbf{r}_0; \\ \dot{\mathbf{r}}_*(t_0) = \dot{\mathbf{r}}_C^0 + \mathbf{v}_0 + \boldsymbol{\omega}_C^0 \times (\mathbf{r}_C^0 + \mathbf{r}_0) \end{cases} \tag{58}$$

where

$$R_\downarrow([-\omega] _\downarrow C) = I - \sin [f_\downarrow c^\uparrow 0] \quad h^\sim_\downarrow C / |\mathbf{h}_\downarrow c| + (1 - \cos [f_\downarrow c^\uparrow 0]) \tag{59}$$

and $\mathbf{r}_c$ *is given by (Eq. (47)).*

Remark 3: The problems (54)) and (55)) are coupled because, in general case, the torque $\boldsymbol{\tau}$ depends of the position vector $\mathbf{r}$.

The relative velocity of the translation motion may be computed as:

$$\mathbf{v} = R_{-\omega_C}\dot{\mathbf{r}}_* - \tilde{\omega}_c R_{-\omega_C}\mathbf{r}_* - \frac{e_c|\mathbf{h}_c|\sin f_c(t)}{p_c}\frac{\mathbf{r}_c^0}{r_c^0} \tag{60}$$

This result shows a very interesting property of the translational part of the relative orbital motion problem (55). We have proven that this problem is super-integrable by reducing it to the classic Kepler problem [11, 12]; [31, 32]. The solution of the translational part of the relative orbital motion problem is expressed thus:

$$\mathbf{r} = \mathbf{r}(t, t_0, \mathbf{r}_0, \mathbf{v}_0); \ \mathbf{v} = \mathbf{v}(t, t_0, \mathbf{r}_0, \mathbf{v}_0) \tag{61}$$

The exact closed form, free of coordinate, solution of the translational motion can be found in [11, 12]; [31, 32]; [34].

5. Conclusions

The paper proposes a new method for the determination of the onboard complete solution to the full-body relative orbital motion problem.

Therefor, the isomorphism between the Lie group of the rigid displacements $\mathbb{SE}_3$ and the Lie group of the orthogonal dual tensors $\underline{\mathbb{SO}}_3$ is used. It is obtained a Poisson-Darboux like problem written in the Lie algebra of the group $\underline{\mathbb{SO}}_3$, an algebra that is isomorphic with the Lie algebra of the dual vectors.

Using the above results, the free of coordinate state equation of the rigid body motion in arbitrary non-inertial frame is obtained.

The results are applied in order to offer a coupled (rotational and translational motion) state equation and a representation theorem for the onboard complete solution of full body relative orbital motion problem. The obtained results interest the domains of the spacecraft formation flying, rendezvous operation, autonomous mission and control theory.

Appendix

In this appendix we will present some algebraic properties for dual numbers, dual vectors and dual tensors. More details can be found in [16], [17], [18], [19], [20], [21], [22], [23], [2].

1. Dual numbers

Consider the set of real dual numbers to be denoted by

$$\underline{\mathbf{R}} = \mathbb{R} + \varepsilon\mathbb{R} = \left\{\underline{a} = a + \varepsilon a_0 | a, a_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\right\} \tag{62}$$

where $a = Re(\underline{a})$ is the real part of $\underline{a}$ and $a_0 = Du(\underline{a})$ the dual part. The sum and product between dual numbers generate a ring with zero divisors structure for $\underline{\mathbf{R}}$. Among he many properties of dual numbers, the magnitude and the inverse are the ones mostly used in this paper. The magnitude of a dual number fulfils $|\underline{a}|^2 = \underline{a}^2$ and can be computed using $|\underline{a}| = |a| + \varepsilon \operatorname{sgn}(a) a_0$, while ist inverse, denoted by $\underline{a}^{-1} \in \underline{\mathbf{R}}$, exists if and only if $Re(\underline{a}) \neq 0$ and is computed using $\underline{a}^{-1} = \frac{1}{\underline{a}} = \frac{1}{a} - \varepsilon \frac{a_0}{a^2}$. Also, $\underline{a} \in \underline{\mathbf{R}}$ is a zero divisor if and only if $Re(\underline{a}) = 0$. Based on these properties results that $(\underline{\mathbf{R}}, +, \cdot)$ is a commutative and unitary ring and any element $\underline{a} \in \underline{\mathbf{R}}$ is either invertible or zero divisor.

Any differentiable function $f: S \subset \mathbb{R} \to \mathbb{R}, f = f(a)$ can be completely defined on $\underline{S} \subset \underline{\mathbf{R}}$ such that:

$$f: \underline{S} \subset \underline{\mathbf{R}} \to \underline{\mathbf{R}}; f(\underline{a}) = f(a) + \varepsilon a_0 f'(a) \tag{63}$$

Based on the previous property, two of the most important functions have the following expressions: $\cos\underline{a} = \cos a - \varepsilon a_0 \sin a; \sin\underline{a} = \sin a + \varepsilon a_0 \cos a$; $\operatorname{artan}\underline{a} = \operatorname{artan} a + \varepsilon \frac{a_0}{1 + a^2}$.

2. Dual vectors

In the Euclidean space, the linear space of free vectors with dimension 3 will be denoted by $\mathbf{V}_3$. The ensemble of dual vectors is defined:

$$\underline{\mathbf{V}}_3 = \mathbf{V}_3 + \varepsilon\mathbf{V}_3 = \left\{\underline{\mathbf{a}} = \mathbf{a} + \varepsilon\mathbf{a}_0;\ \mathbf{a}, \mathbf{a}_0 \in \mathbf{V}_3, \varepsilon^2 = 0, \varepsilon \neq 0\right\} \tag{64}$$

where $\mathbf{a} = Re(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = Du(\underline{\mathbf{a}})$ the dual part. For dual vectors, three products are considered: scalar product (denoted by $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product (denoted by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$), and triple scalar product (denoted by $(\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}) = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$). Regarding algebraic structure, $(\underline{\mathbf{V}}_3, +, \cdot_{\underline{\mathbb{R}}})$ is a free $\underline{\mathbb{R}}$-module [18].

The magnitude of $\underline{\mathbf{a}}$, denoted by $|\underline{\mathbf{a}}|$, is a dual number which fulfills $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be computed using

$$|\underline{\mathbf{a}}| = \begin{cases} \|\mathbf{a}\| + \varepsilon \dfrac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}, Re(\underline{\mathbf{a}}) \neq 0 \\ \varepsilon \|\mathbf{a}_0\|, Re(\underline{\mathbf{a}}) = 0 \end{cases} \tag{65}$$

where $\|\cdot\|$ is the Euclidean norm. For any dual vector $\underline{\mathbf{a}} \in \underline{\mathbf{V}}_3$, if $|\underline{\mathbf{a}}| = 1$ then $\underline{\mathbf{a}}$ is called unit dual vector.

Theorem 11 *For any* $\underline{\mathbf{a}} \in \underline{\mathbf{V}}_3$, *a dual number* $\underline{\alpha} \in \underline{\mathbb{R}}$, *and a unit dual vector* $\underline{\mathbf{u}}_\alpha \in \underline{\mathbf{V}}_3$ *exist in order to have*

$$\underline{\mathbf{a}} = \underline{\alpha}\, \underline{\mathbf{u}}_\alpha \tag{66}$$

The computational formulas for $\underline{\alpha}$ *and* $\underline{\mathbf{u}}_\alpha$ *are* $\pm\underline{\alpha} = |\underline{\mathbf{a}}|$

$$\pm\underline{\mathbf{u}}_\alpha = \begin{cases} \dfrac{\mathbf{a}}{\|\mathbf{a}\|} + \varepsilon \dfrac{\mathbf{a} \times (\mathbf{a}_0 \times \mathbf{a})}{\|\mathbf{a}\|^3}, Re(\underline{\mathbf{a}}) \neq 0 \\ \dfrac{\mathbf{a}_0}{\|\mathbf{a}_0\|} + \varepsilon \mathbf{v} \times \dfrac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall \mathbf{v} \in \mathbf{V}_3, Re(\underline{\mathbf{a}}) = 0 \end{cases} \tag{67}$$

Also, for $Re(\underline{\mathbf{a}}) \neq 0$, $\underline{\alpha}$ and $\underline{\mathbf{u}}_\alpha$ are unique up to a sign change.

3. Dual tensors

A Euclidean dual tensor represents a $\underline{\mathbb{R}}$-linear application of $\underline{\mathbf{V}}_3$ into $\underline{\mathbf{V}}_3$, where:

$$\underline{T}(\underline{\lambda}_1 \underline{\mathbf{v}}_1 + \underline{\lambda}_2 \underline{\mathbf{v}}_2) = \underline{\lambda}_1 \underline{T}(\underline{\mathbf{v}}_1) + \underline{\lambda}_2 \underline{T}(\underline{\mathbf{v}}_2), \forall \underline{\lambda}_1, \underline{\lambda}_2 \in \underline{\mathbb{R}}, \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\mathbf{V}}_3 \tag{68}$$

From now on, any Euclidean dual tensor will be shortly called dual tensor and $\mathrm{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3)$ will denote the free $\underline{\mathbb{R}}$-module of dual tensors. Any dual tensor $\underline{T} \in \mathrm{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3)$ can be decomposed in $\underline{T} = T + \varepsilon T_0$, where $T, T_0 \in \mathrm{L}(\mathbf{V}_3, \mathbf{V}_3)$ are real tensors. The transposed dual tensor, denoted by $\underline{T}^{\mathrm{T}}$, is defined by

$$\underline{\mathbf{v}}_1 \cdot (\underline{T} \underline{\mathbf{v}}_2) = \underline{\mathbf{v}}_2 \cdot (\underline{T}^{\mathrm{T}} \underline{\mathbf{v}}_1), \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\mathbf{V}}_3 \tag{69}$$

while $\forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \in \underline{\mathbf{V}}_3$, $Re(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3) \neq 0$ the determinant is

$$(\underline{T}\underline{\mathbf{v}}_1, \underline{T}\underline{\mathbf{v}}_2, \underline{T}\underline{\mathbf{v}}_3) = \det \underline{T} (\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3). \tag{70}$$

For any dual vector $\underline{\mathbf{a}} \in \underline{\mathbf{V}}_3$ the associated skew-symmetric dual tensor will be denoted by $\underline{\tilde{a}}$ and will be defined by:

$$\underline{\tilde{a}}\underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \forall \underline{\mathbf{b}} \in \underline{\mathbf{V}}_3 \tag{71}$$

The previous definition can be directly transposed into the following result: for any skew-symmetric dual tensor $\underline{\boldsymbol{A}} \in L(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3), \underline{\boldsymbol{A}} = -\underline{\boldsymbol{A}}^T$, the following a uniquely defined dual vector, denoted $\underline{\mathbf{a}} = vect\underline{\boldsymbol{A}}, \underline{\mathbf{a}} \in \underline{\mathbf{V}}_3$, exists so that $\underline{\boldsymbol{A}}\,\underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \forall \underline{\mathbf{b}} \in \underline{\mathbf{V}}_3$. The set of skew-symmetric dual tensors is structured as a free $\underline{\mathbf{R}}$-module of rank 3 and is isomorph with $\underline{\mathbf{V}}_3$ [18].

References

[1] Condurache D., Burlacu A., *Onboard Exact Solution to the Full-Body Relative Orbital Motion Problem*, AIAA Journal of Guidance, Control, and Dynamics, 2016; 2638-2648, vol. 39, no. 12, DOI: 10.2514/1.G000316

[2] Condurache D., Burlacu A., *On Six D.O.F Relative Orbital Motion Parameterization Using Rigid Bases of Dual Vectors*, Advances in the Astronautical Sciences, 2013; 2293–2312 vol. 150,

[3] Filipe N., Tsiotras P., *Adaptive Model-Independent Tracking of Rigid Body Position and Attitude Motion with Mass and Inertia Matrix Identification using Dual Quaternions*, AIAA Guidance, Navigation, and Control (GNC) Conference, Boston, Massachusetts, 2013., DOI: 10.2514/6.2013-5173

[4] Segal S., Gurfil P., *Effect of Kinematic Rotation-Translation Coupling on Relative Spacecraft Translational Dynamics*, Journal of Guidance, Control, and Dynamics, 2009; 1045–1050; vol. 32, no. 3, DOI: 10.2514/1.39320

[5] Condurache D., *Poisson-Darboux problems's extended in dual Lie algebra*, AAS/AIAA Astrodynamics Specialist Conference, Stevenson, WA, USA, 2017;

[6] Alfriend K., Vadali S., Gurfil P., How J., Breger L., *Spacecraft Formation Flying*, New York: Elsevier, 1999, 227–232 DOI: 10.1016/B978-0-7506-8533-7.00214-1

[7] Carter T. E., *New form for the optimal rendezvous equations near a Keplerian orbit*, Journal of Guidance, Control, and Dynamics, 1990; 183-186; vol. 13, no. 1, DOI: 10.2514/3.20533

[8] Gim D.-W., Alfriend K. T., *State Transition Matrix of Relative Motion for the Perturbed Noncircular Reference Orbit*, Journal of Guidance, Control, and Dynamics, 2003; 956–971, vol. 26, no. 6, DOI: 10.2514/2.6924

[9] Sinclair A. J., Hurtado J. E., Junkins J. L., *Application of the Cayley Form to General Spacecraft Motion*, Journal of Guidance, Control, and Dynamics, 2006; 368–373; vol. 29, no. 2, DOI: 10.2514/1.9910

[10] Yamanaka K., Ankersen F., *New State Transition Matrix for Relative Motion on an Arbitrary Elliptical Orbit*, Journal of Guidance, Control, and Dynamics, 2002; 60–66; vol. 25, no. 1. DOI: 10.2514/2.4875.

[11] Condurache D., Martinusi V., *Kepler's Problem in Rotating Reference Frames. Part 1: Prime Integrals, Vectorial Regularization*, Journal of Guidance, Control, and Dynamics, 2007; 192-200; vol. 30, no. 1. DOI: 10.2514/1.20466

[12] Condurache D., Martinusi V., *Kepler's Problem in Rotating Reference Frames. Part 2: Relative Orbital Motion*, Journal of Guidance, Control, and Dynamics, 2007; 201-213, vol. 30, no. 1. DOI: 10.2514/1.20470

[13] Condurache D., Martinusi V., *Relative Spacecraft Motion in a Central Force Field*, Journal of Guidance, Control, and Dynamics, 2007; 873–876; vol. 30, no. 3. DOI: 10.2514/1.26361

[14] Condurache D., Martinusi V., *A Novel Hypercomplex Solution to Kepler's Problem*, Publications of the Astronomy Department of the Eötvös University (PADEU), 2007; 201–213; vol. 19. ISBN 963 463 557

[15] Condurache D., Martinusi V., *Hypercomplex Eccentric Anomaly in the Unified Solution of the Relative Orbital Motion*, Advances in the Astronautical Sciences, 2010; 281–300; vol. 135. ISSN: 0065-3438

[16] Angeles J., *The Application of Dual Algebra to Kinematic Analysis*, Computational Methods in Mechanical Systems, 1998; 3-32; vol. 161. DOI: . 10.1007/978-3-662-03729-4_1

[17] Condurache D., Burlacu A., *Dual Lie Algebra Representations of the Rigid Body Motion*, AIAA/AAS Astrodynamics Specialist Conference, AIAA Paper, San Diego, 2014; 2014-4347. DOI: 10.2514/6.2014-4347

[18] Condurache D., Burlacu A., *Dual Tensors Based Solutions for Rigid Body Motion Parameterization*, Mechanism and Machine Theory, 2014; 390-412; vol. 74. DOI: 10.1016/j.mechmachtheory.2013.12.016.

[19] Condurache D., Burlacu A., *Recovering Dual Euler Parameters from Feature-Based Representation of Motion*, Advances in Robot Kinematics, 2014; 295–305. DOI: 10.1007/978-3-319-06698-1_31.

[20] Pennestri E., Valentini P. P., *Dual Quaternions as a Tool for Rigid Body Motion Analysis: A Tutorial with an Application to Biomechanics*, The Archive of Mechanical Engineering, 2010; 184-205, vol. LVII. DOI: 10.2478/v10180-010-0010-2.

[21] Pennestri E., Valentini P. P., *Linear Dual Algebra Algorithms and their Application to Kinematics*, Multibody Dynamics: Computational Methods and Applications, 2009; 207-229; vol. 12. DOI: 10.1007/978-1-4020-8829-2_11.

[22] Fischer I., *Dual-Number Methods in Kinematics, Statics and Dynamics*, CRC Press, 1998, 1-9. ISBN 9780849391156

[23] Condurache D., A. Burlacu, *Orthogonal dual tensor method for solving the AX= XB sensor calibration problem*, Mechanism and Machine Theory, 2016; 382–404; vol. 104. DOI: 10.1016/j.mechmachtheory.2016.06.002.

[24] Angeles J., *Fundamentals of Robotic Mechanical Systems*, Springer, 2014.

[25] Tsiotras P., Junkins J. L., H. Schaub, *Higher-order Cayley transforms with applications to attitude representations*, Journal of Guidance, Control, and Dynamics, vol. 20, no. 3, pp. 528-534, 1997.

[26] Vasilescu F. H., *Quaternionic Cayley transform, Journal of Functional Analysis*, 1999; 134-162; vol. 164. MR 2000d:47015

[27] Selig J. M., *Cayley maps for SE(3)*, 12th IFToMM World Congress, Besancon, 2007.

[28] Darboux G., *Lecons sur la Theorie Generale des Surfaces et les Applications Geometriques du Calcul Infinitesimal*, Paris: Gauthier-Villars, 1887, 175-179. ark:/13960/t2h70912j.

[29] Condurache D., Burlacu A., *Fractional Order Cayley Transforms for Dual Quaternions based Pose Representation*, Advances in the Astronautical Sciences, 2016; 1317-1339; vol. 156.

[30] Brodsky V., Shoham M., *Dual Numbers Representation of Rigid Body Dynamics*, Mechanism and Machine Theory, 1999; 693–718; vol. 34, no. 5. DOI: 10.1016/S0094-114X(98)00049-4

[31] Condurache D., Martinusi V., *Foucault Pendulum-Like problems: A Tensorial Approach*, International Journal of Non-Linear Mechanics, 2008; 743–760; vol. 43, no. 8. DOI: 10.1016/j.ijnonlinmec.2008.03.009

[32] Condurache D., Martinusi V., *Exact Solution to the Relative Orbital Motion in Eccentric Orbits*, Solar System Research, 2009; 41–52; vol. 43, no. 1. DOI: 10.1134/S0038094609010043.

[33] Tanygin S., *Attitude parameterizations as higher-dimensional map projections*, Journal of Guidance, Control, and Dynamics, 2012; 13-24; vol. 35, no. 1. DOI: 10.2514/1.54085.

[34] Condurache D., Martinusi V., *Quaternionic Exact Solution to the Relative Orbital Motion Problem*, Journal of Guidance, Control, and Dynamics, 2010; 1035–1047; vol. 33, no. 4. DOI: 10.2514/1.47782.

[35] Condurache D., Martinusi V., *Exact Solution to the Relative Orbital Motion in a Central Force Field*, 2nd International Symposium on Systems and Control in Aeronautics and Astronautics, Shenzhen, PRC, 2008. DOI: 10.1109/ISSCAA.2008.4776296.

[36] Gurfil P., Kasdin J. N., *Nonlinear Modeling of Spacecraft Relative Motion in the Configuration Space*, Journal of Guidance, Control, and Dynamics, 2004; 154–157; vol. 27, no. 1. DOI: 10.2514/1.9343.