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Non-propogation crack criteria and fracture criteria in the case of mixed-mode fracture, taking into account the deterioration

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Abstract. The paper develops in the case of mixed-mode fracture: criteria of non-propagation crack and fracture criteria, for mechanical structures, taking into account the deterioration, namely general relations for the superposition of loadings corresponding to the three fracture modes. It refers to the concepts of stress intensity factor, crack tip opening displacement and J-integral; new fracture criteria taking into account the deterioration, the rate of loading and the scattering of the material characteristics were established. The relations obtained have been compared and verified against experimental data and empirical relations reported in literature. The numerical example allows one to comprehend the practical use of the established relations.

Keywords: mixed-mode fracture deterioration; stress intensity factor; crack tip opening displacement; J – integral; fracture criteria.

1. Introduction

There are numerous cases when a mechanical structure is loaded simultaneously and/or successively with different loads determining normal and/or shear stresses. In many practical cases, either the crack is not perpendicular to the mode I loading direction, or the structure is subjected to multiaxial loading, results in a mixed mode stress field near the crack.

In the papers [1-5] new phenomenological models have been proposed to account for the effects superposition of cracked linear-elastic or nonlinear materials, for the deterioration, for the rate of loading and for the materials characteristics scattering.

• If the mechanical structure *contains cracks*, then, the loading effect due to stresses is superimposed on the effect created by the crack characteristic depth, *a*, using *the*

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Nomenclature

D_i, D_T	- partial deterioration, total deterioration, respectively;
E	- Young's modulus of elasticity;
E_i	- specific energy $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$;
$E_{i,cr}$	- critical value of E_i;
$E(\sigma); E(\tau)$	- specific energy due to normal stress and shear stress, respectively;
G	- shear modulus of elasticity;
J_i	- J – integral $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$;
J_{ic}	- critical value of J_i $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$;
K_i	- stress intensity factor $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$;
K_{ic}	- fracture toughness $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$;
P_{cr}	- critical specific energy participation;
P_i	- specific energy participation of load *i*;
P_T	- total specific energy participation;
Y	- shape factor;
$a;2c$	- characteristic crack depth and crack length, respectively;
t	- time;
$k;k_1$	- exponents;
γ	- shear strain;
δ_i	- crack tip opening displacement $(i = \mathrm{I},\ \mathrm{II},\ \mathrm{III})$;
δ_{ic}	- critical value of δ_i $(i = \mathrm{I},\ \mathrm{II},\ \mathrm{III})$;
ε	- strain;
σ; σ_{cr}	- normal stress and critical normal stress, respectively;
σ_u; σ_y	- ultimate normal stress, yield normal stress;
τ; τ_u	- shear stress; ultimate shear stress, respectively.

stress intensity factor. For materials with linear-elastic behavior and relatively low plastic deformation capacity the stress intensity factors are:

$$K_{\mathrm{I}} = \sigma \cdot Y_{\mathrm{I}} \cdot \sqrt{\pi \cdot a}\ ;\ K_{\mathrm{II,III}} = \tau \cdot Y_{\mathrm{II,III}} \cdot \sqrt{\pi \cdot a}\ . \qquad (1)$$

In relationships (1) Y_{I} and $Y_{\mathrm{II,III}}$ are geometric factors, whose value depend on the structure geometry and on the crack shape. The indexes I, II, and III refer to the fracture modes: opening mode (I), sliding mode (II) and tearing mode (III).

The fracture criteria corresponding to a single fracture mode are $\mathrm{K_i} \geq \mathrm{K_{ic}}$, where $\mathrm{i} = \mathrm{I}; \mathrm{II}; \mathrm{III}$ and $\mathrm{K_{ic}}$ is the fracture toughness.

• Mechanical structures (engines, vessels, ships, aircraft etc…) operate at certain operational parameters (pressure, temperature, speed …), in contact with air, water, corrosive or erosive substances. The cracks, the aging, the embrittlement (due to fast

neutron irradiation, due to hydrogen), as well as fatigue and creep contribute to material *deterioration*.
But structure deterioration may be also due to the superposition of several actions like fatigue and corrosion, fatigue and creep, fatigue and residual stresses, high pressure and high temperature, fatigue and corrosion and creep a.s.o. Deterioration may proceed slowly in time (aging, creep, fatigue) or suddenly (thermal shock, mechanical shock).
It is useful to take into account the deterioration when the strength or the life time of a structure is calculated [6-8].
In the case of a fracture featuring all the three fracture modes, it is necessary to establish the correlation between $\mathrm{K_I}; \mathrm{K_{II}}$ and $\mathrm{K_{III}}$ and to define a suitable fracture criterion. For example, the strain energy release rate for a crack under combined mode I and mode II was used [9]. Many criteria have been suggested to predict mixed mode I+II fracture such as: the maximum tangential stress (MTS) criterion, the maximum energy release rate (G) criterion [10].
For example, by using the MTS criterion for mixed modes I and II, applied to rocks, this criterion predicts greater value for the pure mode I and lower value for pure mode II, comparing to the experimental results; an improving has been obtained with the modified MTS criterion (MMTS) used for mixed-mode fracture prediction [10].
Other researches on this topic were published by [11-15].
Two different energetic criteria have been proposed on the basis of the energy release rate corresponding to mode I (G_I) and mode II (G_{II}) fracture [16], namely linear energetic criterion and quadratic energetic criterion. But how to choose between these two criteria?

2. How we can solve the strength problem of a cracked structure

The stresses at the crack tip are higher than the stresses far away from the crack.
• As to solve a strength problem of a cracked structure one can:
a. calculate the ultimate stress taking into account the shape and the crack dimensions $(a; 2c)$. For example [17-18],

$$\left.\begin{aligned}\sigma_u(a;c) &= \sigma_u \cdot [1 - D_\sigma(a;c)]^{\frac{1}{\alpha+1}};\\ \tau_u(a;c) &= \tau_u \cdot [1 - D_\tau(a;c)]^{\frac{1}{\alpha_1+1}},\end{aligned}\right\} \tag{2}$$

where σ_u, τ_u are the ultimate normal and shear stress respectively, of uncracked sample; $\sigma_u(a;c)$; $\tau_u(a;c)$ – the ultimate normal and shear stress, respectively, of the cracked sample; $D_\sigma(a;c)$; $D_\tau(a;c)$ – the deterioration due to crack by normal stress and shear stress loading. The deterioration $0 \le D \le 1$ is unitless. The exponents $\alpha = 1/k$ and $\alpha_1 = 1/k_1$ becomes from nonlinear, power law, behavior,

$$\sigma = M_\sigma \cdot \varepsilon^k \quad \text{and} \quad \tau = M_\tau \cdot \gamma^{k_1}, \tag{3}$$

where σ; τ is normal and shear stress, respectively; M_σ; M_τ; k and k_1 are material constants. The strength criterion is given by the eq.,

$$\sigma_{eq} \le \sigma_{al}(a;c), \tag{4}$$

where σ_{eq} is the equivalent stress (Tresca or von Mises); $\sigma_{al}(a;c) = \sigma_u(a;c)/c_u$ is the allowable stress and $c_u > 1$ is the safety coefficient;

b. or establishing the conditions (loads and crack) for crack non-propagation. This require a non propagation criterion in the general case of samples/structures with cracks, subjected to a normal stress (σ), an in-plane shear stress (τ) and an uniform out-of-plane shear stress (τ_t) (Fig. 1). In a point near the border of the crack, the stress field is a combination of the opening-mode, sliding-mode and tearing-mode and it is governed by the values of corresponding stress intensity factors K_{I}, K_{II} and K_{III}.

A general dimensionless non-propagation criterion can be written as the following function (Fig. 2),

$$F_{cr}\left(K_{\mathrm{I}}/K_{\mathrm{I}c};K_{\mathrm{II}}/K_{\mathrm{II}c};K_{\mathrm{III}}/K_{\mathrm{III}c}\right) < P_{cr}(t), \tag{5}$$

where $K_{\mathrm{II}c}$; $K_{\mathrm{III}c}$ are the critical values of K_{II} and K_{III}; the term $P_{cr}(t) \le 1$.

In the case of only one fracture mode, the non-propagation criterion is $K_i < K_{ic}$, with $i =$ I, II and III.

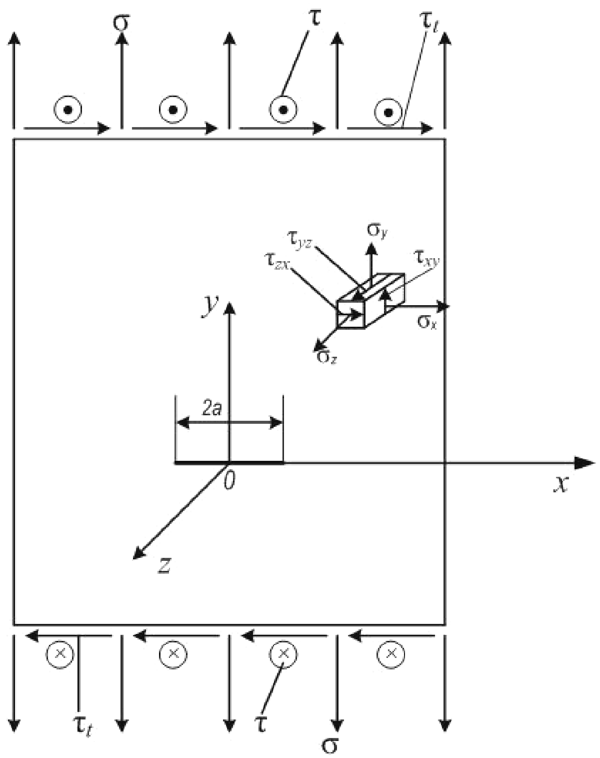

Fig. 1. Simultaneous loading with stresses corresponding to the three fracture modes (I; II; III).

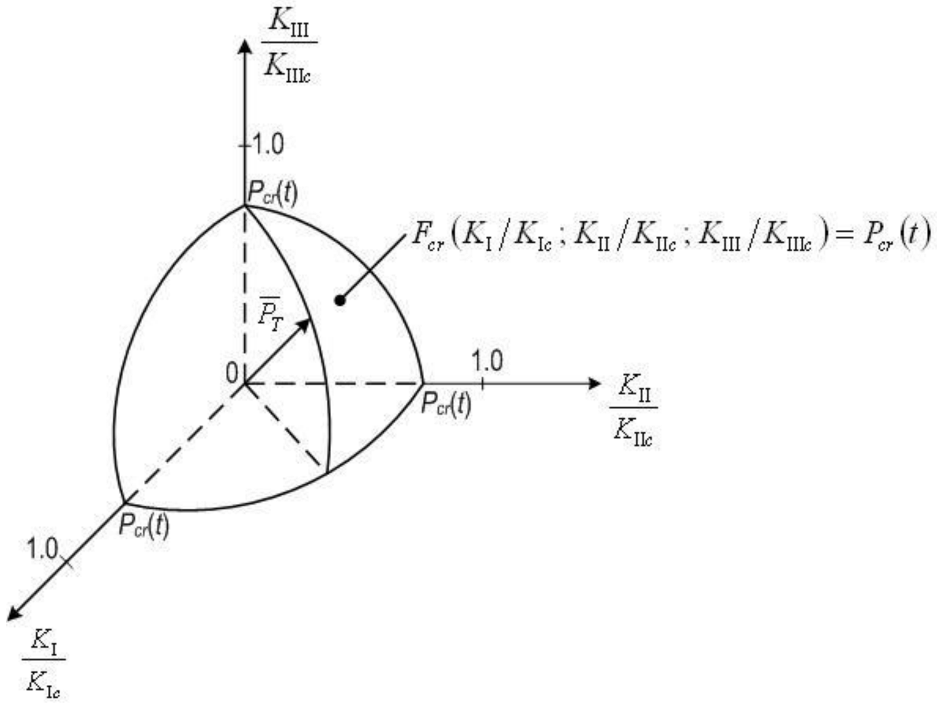

Fig. 2. The surface corresponding to initiation of unstable crack propagation.

• The aim of this paper is to establish a non-propagation criteria (5) in the case of a sample/structure with a crack, subjected to stresses (σ; τ; τ_t) which create a stress field, a combination of the opening – mode, sliding – mode and tearing – mode.
Starting from the specific energy concept, using the principle of critical energy [1;19], mixed-mode failure (I, II and III) criteria are obtained. These criteria take into account the material behaviour, the influence of mechanical characteristics scattering, the rate of loading and the deterioration. The criteria were obtained starting with the concepts of stress intensity factor, crack tip opening displacement and J-integral.

3. Superposition of loadings using the energy concept. A new approach

At a certain point near the border of the crack, one considers that loading takes place simultaneously according to the three fracture modes (I; II and III); the overall load is obtained by adding up the partial loadings. However, this cumulative process or superposition of loads cannot be done by algebraically summing up the stress intensity factors, because the stresses that engendered them behave in a variety of ways ($\sigma \rightarrow \text{opening}\,;\tau \rightarrow \text{sliding}$ and $\tau_t \rightarrow$ tearing). That is why, one resorts to the concept of energy, namely to the concept of *specific energy* (J/kg or J/m^3).
The *participation of the specific energy* – a dimensionless variable – is defined [1] as the ratio between the specific energy introduced by the effective stress (σ or τ) and the critical value corresponding to this stress (σ_{cr} or τ_{cr}),

$$P(\sigma)=\frac{E(\sigma)}{E_{cr}(\sigma)}\cdot\delta_\sigma \text{ and } P(\tau)=\frac{E(\tau)}{E_{cr}(\tau)}, \quad (6)$$

where $E_{cr}(\sigma)$, $E_{cr}(\tau)$ is the critical value of the specific energy due to normal stress and shear stress, respectively;

$$\delta_\sigma = \begin{cases} 1, \text{ if the normal stress, } \sigma, \text{ action is in the sense of the deformation or fracture process;} \\ 0, \text{ if the normal stress has no effect upon the deformation or fracture process;} \\ -1, \text{ if the normal stress, } \sigma, \text{ opposes the deformation or fracture process.} \end{cases} \quad (7)$$

The total participation of the specific energies (unitless) is equal to the sum of the individual participations [20],

$$P_T=\sum_i P_i(\sigma)+\sum_j P_j(\tau).$$

According to the principle of critical energy,
– the criterion of non-propagation crack is,

$$P_T < P_{cr}(t); \quad (8)$$

– the fracture criterion corresponds to the equality,

$$P_T = P_{cr}(t), \quad (9)$$

where t is the time of loading. The total participation of the specific energies in this case is,

$$P_T=\frac{E_{\mathrm{I}}}{E_{\mathrm{I},cr}}\cdot\delta_\sigma+\frac{E_{\mathrm{II}}}{E_{\mathrm{II},cr}}+\frac{E_{\mathrm{III}}}{E_{\mathrm{III},cr}}. \quad (10)$$

Here the total participation of accumulated specific energy (E_{I}, E_{II} and E_{III}) is due to stresses σ, τ and τ_t. The denominators $E_{\mathrm{I},cr}$, $E_{\mathrm{II},cr}$ and $E_{\mathrm{III},cr}$ in relation (10) are the critical values of specific energy E_{I}, E_{II} and E_{III}, respectively.
Each term of relation (10) is dimensionless and represents the fracture participation in the accumulated specific energy.

4. The total participation of the specific energy based on the stress intensity factor

Shear stresses τ and τ_t always act in the sense of the material fracture. When loading occurs according to fracture mode I, stress σ may act in the direction of the crack opening or closing.
The specific energy or density of energy $(E(\sigma)$ and $E(\tau))$ depends on material behavior. In the case of liner-elastic behavior, given by Hooke's law, for uniaxial loading,

$$\sigma = E\cdot\varepsilon \text{ and } \tau = G\cdot\gamma, \quad (11)$$

the specific energies are:

$$E(\sigma) = \frac{\sigma^2}{2E} \text{ and } E(\tau) = \frac{\tau^2}{2G}, \tag{12}$$

and the critical values of specific energies are,

$$E_{cr}(\sigma) = \frac{\sigma_{cr}^2}{2E} \text{ and } E_{cr}(\tau) = \frac{\tau_{cr}^2}{G}. \tag{13}$$

With Eqs. (6), (12) and (13) one obtains the following Eqs. of the specific energy participations,

$$P(\sigma) = \left(\frac{\sigma}{\sigma_{cr}}\right)^2 \cdot \delta_\sigma \text{ and } P(\tau) = \left(\frac{\tau}{\tau_{cr}}\right)^2. \tag{14}$$

Multiplying by $\sqrt{\pi \cdot a}$ $(K_\mathrm{I}$ and $K_{\mathrm{I}cr}$ correspond to the same crack *a*, but different stresses, σ and $\sigma_{cr,}$ respectively) at the numerator and denominator, taking into consideration relations (1), one gets $P(\sigma) \equiv P(K_\mathrm{I})$, or

$$P(K_\mathrm{I}) = \left(\frac{K_\mathrm{I}}{K_{\mathrm{I}cr}}\right)^2 \cdot \delta_\sigma. \tag{15}$$

Analogously, for the shear stresses one gets,

$$P(K_\mathrm{II}) = \left(\frac{K_\mathrm{II}}{K_{\mathrm{II}cr}}\right)^2 \text{ and } P(K_\mathrm{III}) = \left(\frac{K_\mathrm{III}}{K_{\mathrm{III}cr}}\right)^2. \tag{16}$$

From relations (10), (15) and (16), with $K_{\mathrm{I}cr} = K_{\mathrm{I}c}$; $K_{\mathrm{II}cr} = K_{\mathrm{II}c}$ and $K_{\mathrm{III}cr} = K_{\mathrm{III}c}$, one obtains the expression of total participation,

$$P_T = \left(\frac{K_\mathrm{I}}{K_{\mathrm{I}c}}\right)^2 \cdot \delta_\sigma + \left(\frac{K_\mathrm{II}}{K_{\mathrm{II}c}}\right)^2 + \left(\frac{K_\mathrm{III}}{K_{\mathrm{III}c}}\right)^2. \tag{17}$$

Fracture toughness K_{ic} $(i = \mathrm{I}; \mathrm{II}; \mathrm{III})$ is an acknowledged feature of materials. Generally, one has found [21],

$$K_{\mathrm{II}c} = a_\mathrm{II} \cdot K_{\mathrm{I}c} \text{ and } K_{\mathrm{III}c} = a_\mathrm{III} \cdot K_{\mathrm{I}c}, \tag{18}$$

where a_II and a_III are material constants.

For metallic materials a_II may be higher or lesser the unity. For example the steel 90MnCr8V8 is characterized by $a_\mathrm{II} = 1.5$ [21;22]. For nonmetallic brittle materials, rocks for example, the Brazilian disk specimens fracture toughness ratio (K_{IIc}/K_{Ic}) is significantly higher than the theoretical estimations [23].

The investigations, for example, of limestone rock from Saudi Arabia have shown that specimen diameter and crack type have influence on the measured fracture toughness [24]. Mode – I fracture toughness is significantly influenced by specimen diameter and crack type, while their effects on Mode – II fracture toughness are generally negligible. In the case of straight-notched Brazilian disks made of limestone rock from Saudi Arabia the ratio $a_{\mathrm{II}} = K_{\mathrm{IIc}}/K_{\mathrm{Ic}}$ was 2.14 and 2.19 for 84 and 98 mm disks, respectively [24] and 1.09 related to graphite [23]. This is why it is necessary to correlate this ratio with the structure size in the case of rocks, after the specimen standard size has been established. Moreover, there are standard specimens to measure mode I fracture toughness for rocks, but no standard method for determining mode II fracture toughness of rocks [23]; – to establish the value of the ratio K_{IIc}/K_{Ic} is out of the goal of this paper.

To account for elastic – plastic effects, equation (1) will be modified taking into account, for example the Dugdale plastic zone size [25]. The effect of the small plastic zone at the tip of the crack can be corrected by increasing the crack length to include the radius of the plastic zone, r_p. The effective length is $(a + r_p)$ instead of a, such as the stress intensity factor becomes $K_{\mathrm{I}} = Y \cdot \sigma \cdot \sqrt{\pi (a + r_p)}$.

5. A new approach of fracture criterion for mixed-mode loading, based on the stress intensity factor, taking into account the deterioration

In order to obtain the fracture criterion one uses eq. (9). In the case of engineering structures, unlike the case of specimens, $P_{cr}(t)$ should also contain the effect of deteriorations. Fracture occurs when relation (9) is fulfilled, where [26],

$$P_{cr}(t) = \begin{cases} P_{cr}(0) & \text{- for } \textit{samples of undeteriorated} \text{ (virgin) materials;} \\ P_{cr}(0) - D_T(t) & \text{- for deteriorated sample or structures;} \\ P_{cr}(0) - D_T(t) - P_{res} & \text{- for deteriorated sample or structures, having residual stresses.} \end{cases} \tag{19}$$

The deterioration concept $D(t)$ introduced by Kachanov [27] is calculated as a specific energy participation; it is a dimensionless parameter [1].

Deterioration $D(t)$ is comprised between *zero* – for virgin, undeteriorated material and *1.0* – for a deteriorated material (fractured, excessively deformed). The total deterioration,

$$D_T(t) = \sum_i D_i(t),$$

where $D_i(t)$ is the partial deterioration (for the analyzed case, *others than the deterioration due to cracks*).

P_{res} is unitless and introduces the influences of residual stresses [28].

The mechanical characteristics scattering is taken into account through the specific energy participation at $t = 0$,

$$P_{cr}(0)=\begin{cases}1 \text{ – for ideal materials, or in the case of real materials whose mechanical characteristics are mean statistical values (deterministic values);} \\ P_{cr,\min}(0)\ldots P_{cr,\max}(0) \text{ – for real material samples whose mechanical characteristics have a stochastic distribution, where } P_{cr,\max}(0)\le 1 \text{ and } P_{cr,\min}(0)>0.\end{cases} \quad (20)$$

Because $D_T(t)\in[0;1]$, for real structures without residual stresses, $P_{cr}(t)\le 1$.

With Eqs. (9) and (17), it results the following *general fracture criterion for linear-elastic materials, as a result of loading superposition, according to the three modes of fracture*, taking into account the *deteriorations*,

$$\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^2\cdot\delta_\sigma+\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}}\right)^2+\left(\frac{K_{\mathrm{III}}}{K_{\mathrm{III}c}}\right)^2=P_{cr}(t). \quad (21)$$

Eq. (21) corresponds to a specimen *statically loaded*. This Eq. describes the surface corresponding to the initiation of unstable crack propagation in a linear-elastic material (Fig. 2).

By the same procedure for a nonlinear, power law, behavior (3), one obtains [1],

$$\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^{\alpha+1}\cdot\delta_\sigma+\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}}\right)^{\alpha_1+1}+\left(\frac{K_{\mathrm{III}}}{K_{\mathrm{III}c}}\right)^{\alpha_1+1}=P_{cr}(t), \quad (22)$$

where $\alpha = 1/k$ and $\alpha_1 = 1/k_1$.

In case the *loads are applied by shock* the exponents are replaced by 1.0 but if the *loading* is *rapid* in relations (21) the exponents are replaced by 1.5 (as reported by Faupel [29]).

Considering a deterministic, mean statistical values of the mechanical characteristics (yield stress, ultimate stress, toughness…), $P_{cr}(0)=1$, such as generally,

$$P_{cr}(t)=1-D(t)-P_{res}. \quad (23)$$

If,

$$\left.\begin{array}{l} P_T < P_{cr}(t) \text{ - loading is subcritical;} \\ P_T = P_{cr}(t) \text{ - loading is critical;} \\ P_T > P_{cr}(t) \text{ - loading is supercritical.}\end{array}\right\} \quad (24)$$

Eqs. (21) superposes the effect of the three modes of failure in the case of linear-elastic behavior of the specimen material; it allows to account for the rate of loading, for the values of mechanical characteristics scattering, as well for the deterioration (other than due to cracks) and residual stresses.

This result is quite different from MTS (maximum tangential stress) and MMTS (modified MTS) criteria used for rocks [10;23;24]. For such brittle materials loaded in compression, the *maximum stress theory* was used to predict failure.
According to [10], after experimental work on marble rock materials it was concluded: “there is a high difference between the experimental and theoretical results suggested by the MTS criterion. In the case of mixed modes, this criterion predicts greater values around the pure mode I and lower values around pure mode II, relative to experimental results”.
On the other side, in [23] it is reported that “the classical MTS criterion” predicts that the fracture toughness ratio K_{IIc} / K_{Ic} for any brittle material is a constant value equal to 0.866, but this value is significantly different from the experimental results”.
The proposed criteria in this paper is in good agreement with many experimental results. But, a challenge may be to replace the Eqs. like (21) with Eqs. based on the critical stresses relied on crack sizes [30].

6. Verification of the proposed general fracture criteria (21) and (22)

6.1. Verification by empirical relations

There are some empirical relations for effects superposition in fracture mechanics. Through these empirical relations, the general fracture criteria (21) and (22) developed in this paper will be verified.
If $P_{cr}(t)=1$ and $\delta_\sigma = 1$ (σ is a tension stress), introducing different values for the exponents, Eq. (22) becomes the empirical relation proposed by Panasyuk [31;32],

$$\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^{n_1} + \left(\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}}\right)^{n_2} + \left(\frac{K_{\mathrm{III}}}{K_{\mathrm{III}c}}\right)^{n_3} = 1, \tag{25}$$

where n_1, n_2 and n_3 are – structural – sensitive parameters determined from the experiment. In comparison with criteria (21) and (22) eq. (25) is a particular case.
The criteria (21) and (22) obtained on the basis of material behavior comprises the effect of the stress σ sign, the effect of mechanical characteristics scattering $(P_{cr}(0))$, *the effect of deterioration* (others than due to crack) and residual stresses.
For an *undeteriorated sample*, without residual stresses, in case σ opens the crack $(\delta_\sigma = 1)$ and $P_{cr}(0)=1$, one obtains the empirical case reported in the paper [21] as fracture criterion.
From the general relation (21) we get:
– for the first two modes of fracture, in the case of linear – elastic behavior,

$$\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^{2} \cdot \delta_\sigma + \left(\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}}\right)^{2} = P_{cr}(t). \tag{26}$$

With $\delta_\sigma = 1$ and $P_{cr}(t)=1$ obtains the relationship reported by [33;34];

– for the first and the third mode of fracture, in the case of linear-elastic behavior, one obtains,

$$\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^2 \cdot \delta_\sigma + \left(\frac{K_{\mathrm{III}}}{K_{\mathrm{III}c}}\right)^2 = P_{cr}(t). \tag{27}$$

If $P_{cr}(t)=1$ and $\sigma > 0$, eq. (27) becomes the relationship reported by [35];

– for the second and the third mode of fracture, it results,

$$\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}}\right)^2 + \left(\frac{K_{\mathrm{III}}}{K_{\mathrm{III}c}}\right)^2 = P_{cr}(t). \tag{28}$$

6.2. Verification by experimental results

Mechanical characteristics generally experience stochastic values which accounts for the critical participation values; it ranges over an interval, $P_{cr}(t) = P_{cr,\min}(t)\ldots P_{cr,\max}(t)$ as one may see in figure 3 and in the papers [33;34;36]. Consequently, even Eq. (26), a particular case of relation (21), is more general than the empirical relation (25), as it takes into consideration the influence of the stochastic distribution of the mechanical characteristics, through the value of $P_{cr}(0)$. The experiments have shown that $P_{cr}(t) \le 1$. The points obtained experimentally for aluminium, aluminium alloy, aluminium composite and stainless steel specimens loaded in fracture modes I and II are located between the curves drawn by relation (26) for $\sqrt{P_{cr}}(t) = 0.9$ and 1.0 (Fig. 3).

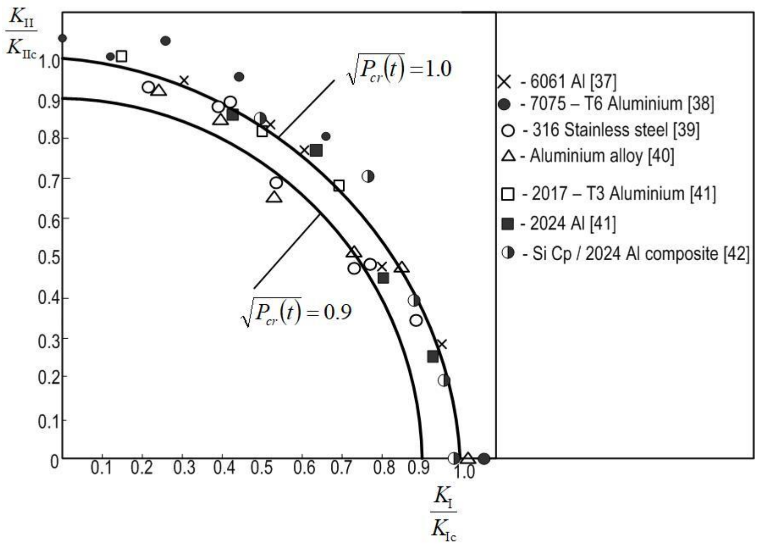

Fig. 3. Comparisons of predicted ((26) with $\sqrt{P_{cr}(t)} = 0.9$ and 1.0) and experimental results of loads superposition in the case of mode I and II of fracture, when σ opens the crack $(\delta_\sigma = 1)$.

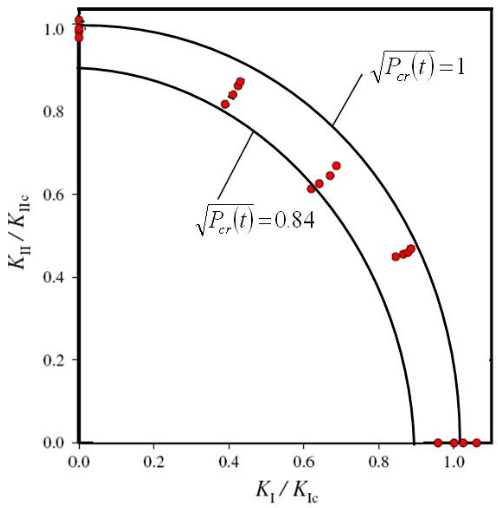

Fig. 4. Experimental results in the case of a polymetilmetacrilate sample under mixed-mode I and II loading (reported as K_{II}/K_{Ic} versus K_I/K_{Ic} in the paper [43], represented here as K_{II}/K_{IIc} versus K_I/K_{Ic} (● – experimental points; the curves are drawn by relation (26) with $\sqrt{P_{cr}(t)}=1.0$ and 0.84).

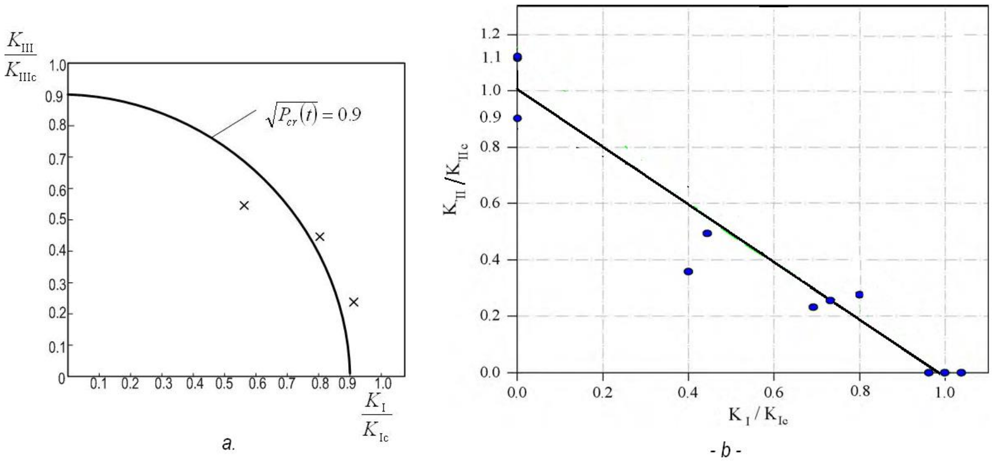

Fig. 5. *a.* Comparisons of predicted (27) and experimental results of effects superposition [44]. *b.* prediction of mixed mode fracture resistance for Neiriz marble rock on the basis of criterion (Eq. (29)) proposed in this paper. Points (•) are test data [10]).

The results of mixed mode I and II fracture experiments in the case of a polymetilmetacrilate (PMMA) specimen, given in figure 4, *a* describes the dependence of K_{II}/K_{Ic}, versus K_I/K_{Ic} [43]. The value in the origin of the ordinate is $K_{II}/K_{Ic} \approx 0.8$. In the ordinate $K_{II}/K_{IIc}=1$ the result is $K_{IIc}=0.8K_{Ic}$ or $K_{Ic}=1.25K_{IIc}$. That means: on the ordinate the experimental points must be translated to 1.25 times the values reported in the paper [43]. One obtains figure 4, which corresponds to relation (26) with $\sqrt{P_{cr}(t)} = 1$ and 0.84.

In the case of the first and third fracture mode for a specimen of mild steel, relation (27) is verified with $\sqrt{P_{cr}(t)} = 0.9$ (Fig. 5, *a*). The experimental points used in figures 4 and 5, *a* have been reported in the paper [44] describing the dependence of $K_{\mathrm{I}}/K_{\mathrm{I}c}$ vs. $K_{\mathrm{II}}/K_{\mathrm{I}c}$ and $K_{\mathrm{I}}/K_{\mathrm{I}c}$ vs. $K_{\mathrm{III}}/K_{\mathrm{I}c}$ instead of $K_{\mathrm{I}}/K_{\mathrm{I}c}$ vs. $K_{\mathrm{II}}/K_{\mathrm{II}c}$ and $K_{\mathrm{I}}/K_{\mathrm{I}c}$ vs. $K_{\mathrm{III}}/K_{\mathrm{III}c}$ as in the present paper.

For marble rock the critical shear stress in front of crack tip is considered a constant material property; it depends on stress intensity factors K_I and K_{II} [10]. As to predict the mixed mode fracture of such brittle materials the maximum tangential stress (MTS) criterion and the modified maximum tangential stress (MMTS) criterion have been used.

As to illustrate the dependence K_{II}/K_{IIc} vs. K_I/K_{Ic}, one calculates from paper [10] the mean value on the ordinate, $K_{II}/K_{Ic} \approx 1.162$. Because in the origin of the ordinate $K_{II}/K_{IIc} = 1$, it results $K_{IIc} = 1.162\ K_{Ic}$ or $K_{Ic} = 0.8606\ K_{IIc}$. That means: on the ordinate K_{II}/K_{IIc} the experimental points must be translated to 0.8606 times the values reported. One obtains figure 5, *b* where the line drawn corresponds to $P_{cr}(t) = 1$ and to Eq. (21) in the case of loads applied by shock, namely,

$$\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}} + \frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}} = 1. \tag{29}$$

It seems to be in better agreement with experimental results than MTS and MMTS criteria do. This comparison need extended researches.

The curves in Figures 6, drawn on the basis of the general Eq. (21) show a good agreement with the experimental data reported in literature.

The experimental data from literature systematized by [45], were changed as in Figures 4 and 5, considering the:

– ratio $K_{\mathrm{II}}/K_{\mathrm{II}c}$ instead of $K_{\mathrm{II}}/K_{\mathrm{I}c}$ (Fig. 6, *a*);

– ratio $K_{\mathrm{III}}/K_{\mathrm{III}c}$ instead of $K_{\mathrm{III}}/K_{\mathrm{I}c}$ (Fig. 6, *b*);

– ratios $K_{\mathrm{II}}/K_{\mathrm{II}c}$ and $K_{\mathrm{III}}/K_{\mathrm{III}c}$ instead of $K_{\mathrm{II}}/K_{\mathrm{I}c}$ and $K_{\mathrm{III}}/K_{\mathrm{I}c}$, respectively (Fig. 6, *c*).

The curves in Figure 6 have been drawn with the particular Eqs. (26) – (28), taking $\delta_\sigma = 1$ and $P_{cr}(t) = 1$.

For *uniaxial compressive* applied *stress* for a glass plate specimen with an inclined crack, some experimental values have been reported by [48]. Figure 7, *a* shows the experimental results of the $K_{\mathrm{II}}/K_{\mathrm{II}c} \div K_{\mathrm{I}}$ locus for compressive applied stresses of

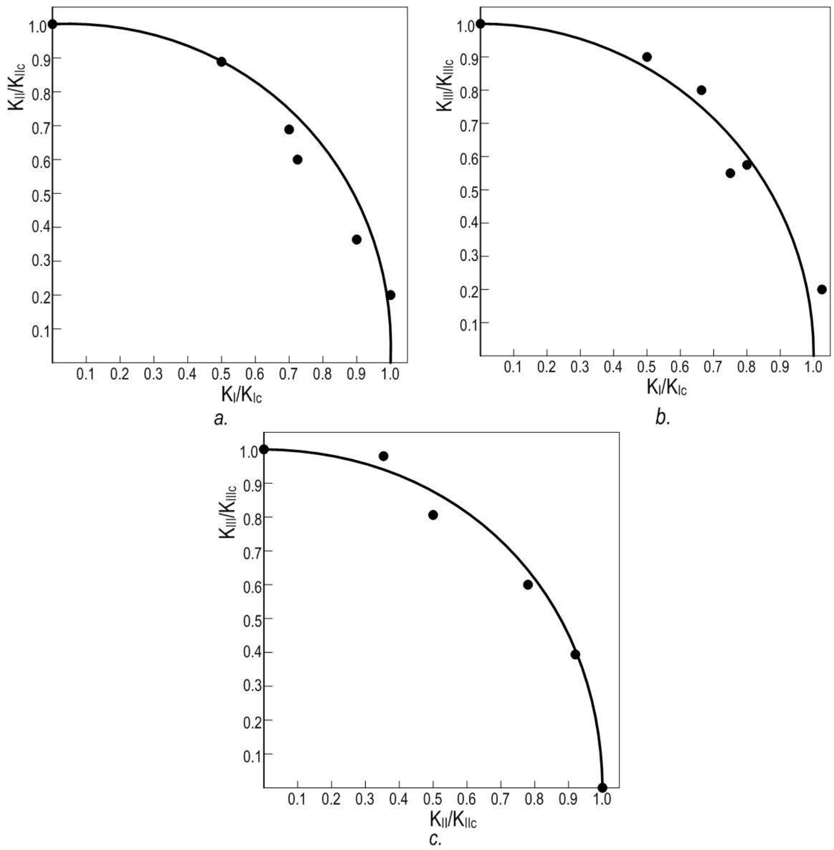

Fig. 6. *a.* Fracture curve for mixed mode I and II based on eq. (26). The experimental data from [46]; *b.* fracture curve for mixed mode I and III based on eq. (27). The experimental data from [46;47]; *c.* fracture curve for mixed mode II and III based on eq. (28). The experimental data from [46;47].

cracked glass. The $K_{\mathrm{II}}/K_{\mathrm{II}c} \div K_{\mathrm{I}}$ curve for compression is basically different from that in tension. A crack under mode – I does not extend in compression, such as the curve does not intersected the K_1 – axis [48]. Because glass is a brittle material, in this case $K_{\mathrm{III}} \equiv 0,$ from relation (21) or (26) one obtains,

$$\frac{K_{\mathrm{II}}}{K_{\mathrm{II}c}} = \left[1 + \left(\frac{K_{\mathrm{I}}}{K_{\mathrm{I}c}}\right)^2\right]^{0.5}, \tag{30}$$

where $P_{cr}(t)$ was replaced by $P_{cr}(t) = 1$ and $\delta_\sigma = -1$ because the stress $(\sigma < 0)$ in this case closes the crack.

Figure 7, *a* shows the increase of the ratio $K_{\mathrm{II}}/K_{\mathrm{IIc}}$ when $\sigma < 0$ and $K_{\mathrm{I}} < 0$. This experimental result is in accordance with the theoretical one given by Eq. (30).

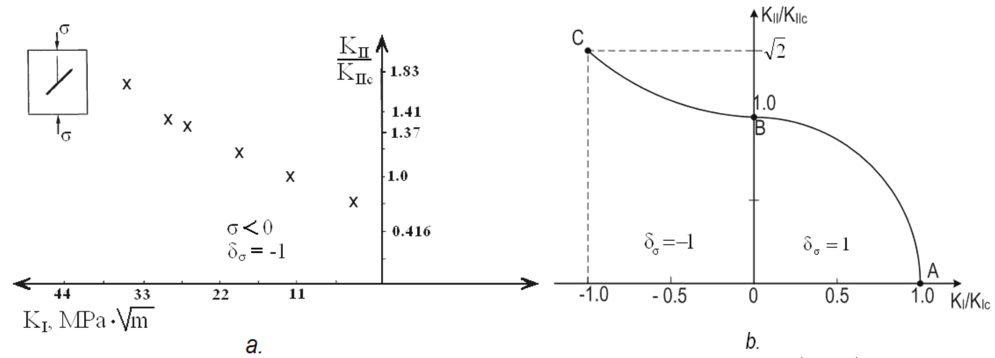

Fig. 7. *a.* The increase of the ratio K_{II}/K_{IIc} with K_I for compression normal stress $(\sigma < 0)$ in cracked glass (processed after [48]); *b.* the curve drawn by relation (26) for $\delta_\sigma = 1$ and -1.

Figure 7, *b* shows graphically the relation (26) where $P_{cr}(t) = 1$. Curve *AB* corresponds to tensile stress σ, which opens the crack $(\delta_\sigma = 1)$, while curve BC corresponds to compressive stress σ that closes the crack $(\delta_\sigma = -1)$. The maximum value of the $K_{II}/K_{IIc} = \sqrt{2}$ is obtained for $|K_I / K_{Ic}| = 1$, according to relation (30).

7. A fracture criterion for mixed – mode loading, based on the crack tip opening displacement (CTOD), taking into account the deterioration

Starting with the correlation established by [49] for structural assessment for fully plastic conditions, it has been obtained [17],

$$P(\delta_I) = \left(\frac{\delta_I}{\delta_{Ic}}\right)^{k+1}, \tag{31}$$

which was verified with the experimental data reported by [50].
Similar Eqs. may be written for the participations of specific energies corresponding to δ_{II} and δ_{III}. The total participation becomes,

$$P(\delta) = \left(\frac{\delta_I}{\delta_{Ic}}\right)^{k+1} + \left(\frac{\delta_{II}}{\delta_{IIc}}\right)^{k_1+1} + \left(\frac{\delta_{III}}{\delta_{IIIc}}\right)^{k_1+1}, \tag{32}$$

where δ_{Ic}, δ_{IIc} and δ_{IIIc} are the critical values of δ_I, δ_{II} and δ_{III}.
The criterion of crack propagation, as a result of Eqs. (9) and (32), is,

$$\left(\frac{\delta_I}{\delta_{Ic}}\right)^{k+1} + \left(\frac{\delta_{II}}{\delta_{IIc}}\right)^{k_1+1} + \left(\frac{\delta_{III}}{\delta_{IIIc}}\right)^{k_1+1} = P_{cr}(t). \tag{33}$$

In the case of linear-elastic behaviour the exponents become $k + 1 = k_1 + 1 = 2$.
The following empirical Eq. has been proposed by [51],

$$\left(\frac{\delta_{\mathrm{I}}}{\delta_{\mathrm{I}c}}\right)^{n_0}+\left(\frac{\delta_{\mathrm{II}}}{\delta_{\mathrm{II}c}}\right)^{t_0}+\left(\frac{\delta_{\mathrm{III}}}{\delta_{\mathrm{III}c}}\right)^{k_0}=1, \tag{34}$$

where the exponents were not defined and does not consider the scattering of the critical values of CTOD. In conclusion, the Eq. (33) obtained in the paper based on a physical and a mathematical support, is beyond the empirical eq. (34).

8. A fracture criterion for mixed – mode loading based on the J – integral concept, taking into account the deterioration

From Eqs. (9) and (10), in the case $\delta_{\sigma}=1\ (\sigma>0)$ obtains,

$$\frac{J_{\mathrm{I}}}{J_{\mathrm{I}c}}+\frac{J_{\mathrm{II}}}{J_{\mathrm{II}c}}+\frac{J_{\mathrm{III}}}{J_{\mathrm{III}c}}=P_{cr}(t), \tag{35}$$

where J_i (with $i=$ I; II; III) is the J – integral, while J_{ic} is the critical value of the J_i – integral. Because J – integral is the rate of change of potential energy with respect to an incremental extension of the crack, Eq. (35) is a sum of energies ratios at a power equal with unity.

9. Numerical example

The relations deduced in the paper have been verified by experimental data reported in literature, as well as by empirical relations reported in literature. This is why now these relations can be used for practical calculations.

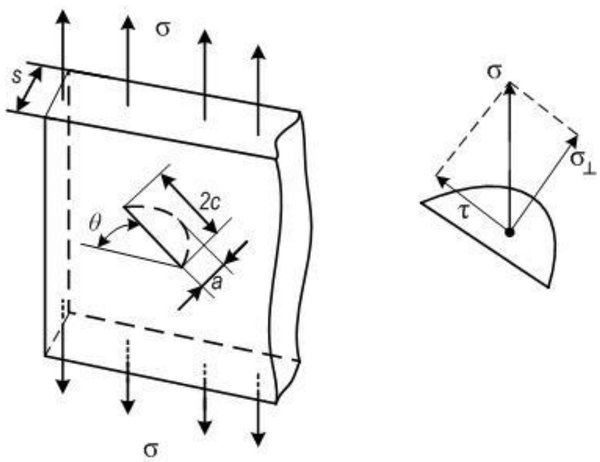

Fig. 8. Crack at angle $\theta=30°$ to the axial direction.

- An elliptical crack in the wall of a steel structure (Fig. 8) is at angle $\theta=30^{\circ}$ to the axial direction. The crack features $a=4\,\mathrm{mm}$ and $c=8\,\mathrm{mm}$. Let us check whether the crack is harmless under stress σ = 60 MPa loading, if it is loaded in linear-elastic field.

The material of structure is a steel featuring: – ultimate stress, $\sigma_u = 510\text{ MPa}$; – yield stress, $\sigma_y = 390\text{ MPa}$; – number of cycles at the knee point, $N_0(\sigma) = 2 \times 10^6$ cycles; – exponents: $m = 3$ taken from Basquin's law [52],

$$\sigma_a^m \cdot N = \text{constant}, \tag{36}$$

where σ_a is the normal stress amplitude; N – the number of cycles up to failure. There are no residual stresses $(P_{res} = 0)$

The wall thickness $s = 20\text{ mm}$, $K_{Ic} = 81\text{ MPa} \cdot \text{m}^{0.5}$ and $K_{IIc} = 70.148\text{ MPa} \cdot \text{m}^{0.5}$.

The structure has been before fatigue loaded by $n = 5 \times 10^4$ cycles with a fully reversed normal stress amplitude at which the number of cycles to failure $N_\sigma = 8 \times 10^5$ cycles.

Solution. Stress σ is projected onto the crack plane (τ) and perpendicular to the crack plane $(\sigma_\perp)$, such as,

$$\sigma_\perp = \sigma \cdot \cos^2\theta = 60 \cdot \cos^2 30° = 45\text{ MPa};$$

$$\tau = \sigma \cdot \sin\theta \cdot \cos\theta = 60 \cdot \sin 30° \cdot \cos 30° = 25.98\text{ MPa}.$$

The corresponding stress intensity factors are:

$$K_\text{I} = \sigma_\perp \cdot Y \cdot \sqrt{\pi \cdot a} = 45 \times 1.211 \cdot \sqrt{\pi \times 4 \times 10^{-3}} = 61.08\text{ MPa} \cdot \text{m}^{0.5};$$

$$K_\text{II} = \tau \cdot Y' \cdot \sqrt{\pi \cdot a} = 25.98 \times 1.211 \cdot \sqrt{\pi \times 4 \times 10^{-3}} = 35.27\text{ MPa} \cdot \text{m}^{0.5},$$

where one considers $Y' \approx Y$ and $Y = \dfrac{1.1}{\Phi^2} = 1.0656$. For $a/c = 4/8 = 0.5$ results $\Phi = 1.211$ [53].

The total energy participation which corresponds to relationship (17) is $(K_\text{III} = 0$ and $\delta_\sigma = 1$, because $\sigma > 0)$,

$$P_T = \left(\frac{\text{K}_\text{I}}{\text{K}_\text{Ic}}\right)^2 \cdot \delta_\sigma + \left(\frac{\text{K}_\text{II}}{\text{K}_\text{IIc}}\right)^2 = \left(\frac{61.08}{81}\right)^2 \times 1 + \left(\frac{35.27}{70.148}\right)^2 = 0.8214.$$

The deterioration produced by the before fatigue loading [3; 8],

$$D(n) = \left(\frac{n}{N(\sigma)}\right)^{\frac{2}{m}} = \left(\frac{5 \times 10^4}{8 \times 10^5}\right)^{\frac{2}{3}} = 0.1575.$$

The critical participation (23),

$$P_{cr}(t) = 1 - D_T(t) = 1 - 0.1575 = 0.8425.$$

Because $P_T < P_{cr}(t)$ the loading is subcritical. The crack does not propagate!

10. Summary and conclusions

The calculus of engineering structures based on fracture mechanics concepts is analyzed. For simultaneous loading with stresses corresponding to two or three fracture modes, the empirical relationships found in literature do not take into consideration the behavior of the structure's material, the rate of load, the deterioration and the scattering of the mechanical characteristics of the material.
With the *principle of critical energy*, general fracture criteria for linear-elastic behavior (brittle or quasi-brittle materials) and nonlinear behavior, mixed – mode loaded based on the *stress intensity factor concept* have been derived ((21) and (22)) as well as some particular cases, in accordance with the three known fracture modes ((26) -(28)).
Based on the *crack tip opening displacement concept* the general fracture criterion (33) have been deduced. The deduced criteria may be used for specimens as well as for engineering structures, taking into account the loading rate, the deterioration and the scattering of the mechanical characteristics of the material.
A similar fracture criterion for mixed-mode loading has been obtained starting from the J-integral concept (35).
Residual stresses have an impact, for example, on the welding fracture resistance, namely, on its fatigue strength [54÷56]. In the paper the influence of residual stress is comprised in the eqs. of critical participations (19) and (21).
The obtained relationships are verified against empirical relations and experimental data reported in literature; they are in good agreement with the experimental data.
The numerical example shows the novelty and advantage of the relationships proposed in the paper.
In consequence, the eqs. derived in the paper, taking into account the influence of deterioration and of residual stress about strength of materials seems to be of high degree of generality.

References

[1] Jinescu V.V., *Principiul energiei critice şi aplicaţiile sale*, Editura Academiei Române, Bucureşti, 2005.
[2] Jinescu, V.V., *Cumulation of Effects in calculation the Deterioration of Fatigue loaded Structures*, Int. J. Damage Mech., **21**, 2012, p. 671-695.
[3] Jinescu V.V., Iordăchescu V.I., *Calculation of deterioration due to cracks in tubular specimens*, UPB Scientific Bulletin, Series D, 77, 2014, p. 49-160.
[4] Jinescu V.V., Nicolof V-I, Teodorescu N., *Effects Superposition under Imposed Deterioration and Simultaneous Fatigue in Different Regimes*., Int. J. Damage Mech., **26**, 2017, p. 633-650.
[5] Jinescu V.V., Nicolof V-I, Chelu, A., Manea S.-E, *Calculation of the local critical state taking into account the deterioration and the residual stresses*, J Eng. Sci. and Innovation, **2**, 2017, p. 9-21.
[6] Jinescu V.V., Nicolof V.I., Jinescu G., Enachescu G.L., *Unitary approach of mechanical structures and living organisms lifetime*, Rev. Chim. (Bucharest), **67**, 9, 2016, p. 1673-1679.
[7] Jinescu V.V., Teodorescu N., Jinescu G., Panait I.C., *The results of superposition of different actions upon the mechanical structures and living bodies*, Rev. Chim. (Bucharest), **67**, 11, 2016, p. 2607-2013.
[8] Jinescu V.V., *Fatigue Life Prediction for Simultaneous Cyclic Loading with Blocks of Normal Stresses and Shear Stresses*, J Eng. Sci. and Innovation, **1**, 2016, p. 1-16.

[9] Hussain M.A., Pu S.L., Underwood J., *Strain energy release rate for a crack under combined mode I and II fracture analysis*, Fracture Analysis. Proceedings of the 1973 National Symposium on Fracture Mechanics. Part II. ASTM STP 560. Lutherville-Timonium, Md., August:2-28, 1974.
[10] Saghafi H., Ayatollahi M.R, Sistaninia M., *A modified MTS criterion (MMTS) for mixed-mode fracture toughness assessment of brittle materials*, Mat. Sci. & Eng. A527, 2010, p. 5624 - 5630.
[11] Wang W.Z., Buhl P., Klenk A., Liu Y.Z., *Study of creep – fatigue behaviour in a 1000MW rotor using a unified viscoplastic constitutive model with damage*, Int. J. Damage Mech, **25**, 2016, p. 178-202.
[12] Zhu S.-P., Lei Q., Huang H.Z., Yang Y.-J., Peng W., *Mean stress effect correction in strain energy-based fatigue life prediction of metals*, Int. J. Damage Mech, **26,** 2017, p. 1219-1241.
[13] Lin S.-J., Long W., Tian D.-Q., Liao J.-B., *A new fatigue damage accumulation model based on damage equivalence*, Int. J. Damage Mech., **27**, 2018, p. 707-728.
[14] Majdi H.R, Ayatolahy M.R., Torabi A.R., Zaheri A., *Energy – based assessment of brittle fracture in VO-notched polymer specimens under combined compression – shear loading condition*, Int. J. Damage Mech, **28**, 2019, p. 664-689.
[15] Voyiadjis G.Z. and Kattan P.I., *Fundamental aspects for characterization in continuum damage mechanics*, Int. J. Damage Mech., **28**, 2019, p. 200-218.
[16] Chaves F.J.P., de Moura M.F.S.F., da Silva L.F.M., Dillard D.A., *Numerical validation of a crack characterization of bonded joints*, Eng. Fracture Mech., **107**, 2013, p. 38-47.
[17] Jinescu V.V., Nicolof Vali-Ifigenia, *Strength criteria at the tip of the crack*, Eng. Fracture Mech., **142**, 2015, p. 108 – 115.
[18] Jinescu V.V., Nicolof V.I., Chelu A., Manea S. E., *Critical stresses, critical group of stresses and strength of tubular structures without and with cracks*, U.P.B. Sci.Bull., Series D, **77**, 2015, p. 165-176.
[19] Jinescu V.V., *Principiul energiei critice*, Rev. Chim. (Bucharest), **35**, 1984, p. 858-861.
[20] Jinescu V.V., *Stability Determination of Structure under Group of Loads by using the Principle of Critical Energy*, Int.J.Press.Ves.&Piping, **48**, 1991, p. 343-375.
[21] Chang J.H., Jeng J.W., *Calculation of near – tip non – singular stresses under mixed – mode crack surface traction*, Int.J. Fract., **182**, 2013, p. 131-139.
[22] Hiese W., *Gültigkeitskriterien zur Bestimmung von Scherbruchzähigkeiten*, Dissertation, Fakultät für Maschinenbau, Ruhr – Universität Bochum, 1998.
[23] Ayatollahi M., Sistaninia M., *Model II fracture study of rocks using Brazilian disk specimens*, Int. J. Rock Mech. & Mining Sci., **48**, 2011, p. 819 - 826.
[24] Khan K., Al-Shayea N.A., *Effect of specimen geometry and testing method on mixed I-II fracture toughness of a limestone rock from Saudi Arabia*, Rock Mech. Rock Engng., 33, 3, 2000, p. 179 - 206.
[25] Perez N., *Fracture Mechanics*, New York, Boston, Kluwer Academic Publishers, 2004.
[26] Jinescu V.V., *Energia, Energonica şi Termodinamica*, Editura AGIR, Bucureşti, 2016.
[27] Kachanov, L.M., *Time of the Rupture Process under Creep Conditons*, Izv. Akad. Nauk SSSR (Otd. Techn. Nauk.), 8, 1968, p. 26-31.
[28] Jinescu, V.V., *Critical Energy Approach for the Fatigue life Calculation under Blocks with different normal Stress Amplitudes*, Int. J. Mechanical Sci., **67**, 2013, p. 78-88.
[29] Faupel J.H., *Engineering Design*, New York, John Wiley, 1964.
[30] Jinescu V.V., Iordăchescu V.I., Teodorescu N., *Relations for the calculation of critical Stress in Pressure Equipment with Cracks*, Revista de Chimie, 64, 8, 2013, p. 858-863.
[31] Panasyuc V.V., *Some urgent problems of the strength of materials and durability of structures*, Mater Sci., **2**, 2009, p. 5-22.
[32] Panasyuc V.V., Ivanytskyi Ya, Hembara O., *Assessment of hydrogen effect on fracture resistance under complex mode loading*, Eng Fract Mech., **83**, 2012, p. 54 -61.
[33] Shah R.C., Fract. Anal, ASTM STP. 560, 1971, p. 29.
[34] Shah R.C., Fract.Mech. 3, 1974, p. 205.
[35] Shah R.C., *Fracture under combined modes in 4340 steel, in Fracture analysis*, Proceedings of the 1973 National Symposium on Fracture Mechanics, Part II, ASTM STP 560, Lutherville-Timonium, Md., August 29-52, 1974.

[36] Pook L.P., Fract. Mech, **3**, 1971, p. 205.
[37] Wang D.Z., Ye F., Liu J., Yao C.K., Wang Z.G., Yu W.C., *Fatigue crack propagation of extruded 19V – SI Cw/6061 Al composite under mixed-mode (I+II) loading*, Scripta Metall Matter, **32**, 1995, p. 1637-1642.
[38] Otsuka A., Tohgo K., *Fatigue crack initiation and growth under mixed-mode loading in aluminum alloys 2017-T3 and 7075-T6*, Eng. Fracture Mech., **28**, 1987, p. 721-732.
[39] Gao H., Brown M.W., Miller K.J., *Mixed-mode threshold stress intensity factors*, Fatigue Engng. Mater. Struct., **5**, 1982, p. 1-17.
[40] Soh A.K., Bian L.C., *Mixed-mode fatigue crack growth criteria*, Int.J.Fatigue, **23**, 2001, p. 427-439.
[41] Pook L.P., *The fatigue crack direction and threshold behavior of mild steel under mixed-mode I and III loading*, Int.J.Fatigue, **7**, 1985, p. 21-30.
[42] Liu P., Wang Z., *Mixed – mode I and II threshold stress intensity factor and crack deflection angle in SiCp/2024 Al composite*, Scripta Mater., **34**, 1996, p. 1323-1330.
[43] Ayatollahi M.R., Aliha M.R.M., *Analysis of a new specimen for mixed mode fracture tests on brittle materials*, Eng. Fracture Mech., **76**, 2009, p. 1563 – 1573.
[44] Liu Y., Mahadevan S., *Threshold intensity factor and crack growth rate prediction under mixed –mode loading*, Eng. Fract. Mech., **74**, 2007, p. 332-345.
[45] Ayatollahi M.R., Saboori B., *Maximum tangential strain energy density criterion for general mixed-mode I/II/III brittle fracture*, Int. J. Damage Mech., **24**, 2015, p. 263-278.
[46] Yishu Z., *A strain energy criterion for mixed mode crack propagation*, Eng. Fract. Mech., **26**, 1987, p. 533 – 539.
[47] Tingshi Z., Yishu Z., Yanhan W. et al., E*xperimental study on mixed mode fracture criterion*, Huazong Industrial Institute, **1**, 1985, p. 47 – 50.
[48] Gdoutos E.E., *Fracture Mechanics*, An Introduction (Second Edition), Springer, Dordrecht, The Netherlands, 2005.
[49] Schwalbe K.-H. *Introduction of* δ_5 *as an Operational Definition of the CTOD and its Practical Application*, Fracture Mechanics: 26th Volume, ASTM STP 1256, WG Reuter, J.H. Underwood and J.C. Newman Eds., ASTM International, West Conshohocken, PA, 1997, p. 763-778.
[50] Jiliukas A.Iu., *Razruşenie konstrukţionnîh elementov*, Mokslas, Vilnius, 1989.
[51] Panasyuc V.V., *Deformaţionnîe kriterii v mehanike razruşenia*, ФХММ**, 7**, 986, p. 7-17.
[52] Basquin O.H., *The exponent low of endurance tests*, Proc. Am. Soc. Test Mater., **10**, 1910, p. 625-630.
[53] Harvey J.F., *Theory and Design of Pressure Vessels*, (Second Edition), New York, Reinhold Comp, 1974.
[54] Gadallah R., Osawa N., Tanaka S., Tsutsumi S., *A novel approach to evaluate mixed-mode SIFs for a through-thickness crack in a welding residual stress field using an effective welding simulation method*, Eng. Fract. Mech., **197,** 2018, p. 48 – 65.
[55] Zerbst U., Madia M., Vormwald M., Beier H.Th., *Fatigue strength and fracture mechanics – A general perspective*, Eng. Fract. Mech., **198**, 2018, p. 2-23.
[56] Hensel J., Nitschke-Pagel T., Tchoffo Ngoula D., Beier H.-Th., Tchuindjang D., *Welding residual stresses as needed for the prediction of fatigue crack propagation and fatigue strength*, Eng. Fract. Mech., **198**, 2018, p. 123-141.