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The constraint functions of the elements of mechanisms

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Abstract: In the present paper we purpose a discussion concerning the constraints which may appear at one arbitrary element of a mechanism. We also determine the corresponding constraint functions. The obtained results may be applied in practical situations. An application highlights the theory.

Keywords: constraint, function of constraint, mechanism, multibody

1. Introduction

It is well known that in the multibody approach for the rigid solid dynamics, the equation of motion may be written as

$$\begin{bmatrix} [\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\boldsymbol{\lambda}\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}\} \\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{1}$$

the notations having the significance given in [1].

In this way, the problem reduces to the determination of the matrix of constraints $[\mathbf{B}]$ for an arbitrary rigid body.

In the particular case of the mechanisms, the constraints may appear either from the connections with the external system or from the connections between the elements of the mechanism.

A classical approach of the analysis and synthesis of the mechanisms is presented in [2]. Some applications for simple mechanisms may be found in [3, 4, 5, 6, 7, 8, 9, 10, 11].

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Further on, we will analyze different types of constraints that may appear at mechanisms determining, for each case, the function or functions of constraints.

2. Notations

We will use the following notations:

– $OXYZ$ – the fixed reference system;

– $O_i x_i y_i z_i$ – mobile reference system rigidly linked to the element i of the mechanism;

– ψ_i, θ_i, φ_i – Bryan's rotational angles of the system $O_i x_i y_i z_i$ with respect to the fixed reference system $OXYZ$;

– $[\boldsymbol{\psi}_i]$, $[\boldsymbol{\theta}_i]$, $[\boldsymbol{\varphi}_i]$ – the square matrices given by

$$[\boldsymbol{\psi}_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_i & -\sin\psi_i \\ 0 & \sin\psi_i & \cos\psi_i \end{bmatrix}, \; [\boldsymbol{\theta}_i] = \begin{bmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix}, \quad [\boldsymbol{\varphi}_i] = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{2}$$

– $[\mathbf{A}_i]$ – the rotational matrix of the element i

$$[\mathbf{A}_i] = [\boldsymbol{\psi}_i][\boldsymbol{\theta}_i][\boldsymbol{\varphi}_i]; \tag{3}$$

– X_i, Y_i, Z_i – the coordinates of the point O_i with respect to the fixed reference system;

– P_j, Q_j – points of one element of the mechanism; this points will be defined in each case;

– $\{\mathbf{R}_{O_i}\}$ – the column matrix

$$\{\mathbf{R}_{O_i}\} = [X_i \;\; Y_i \;\; Z_i]^{\mathrm{T}}; \tag{4}$$

– $x_P^{(i)}$, $y_P^{(i)}$, $z_P^{(i)}$ – the coordinates of a point P belonging to the element i with respect to the mobile reference system $O_i x_i y_i z_i$;

– $\{\mathbf{r}_P^{(i)}\}$ – the column matrix

$$\{\mathbf{r}_P^{(i)}\} = [x_P^{(i)} \;\; y_P^{(i)} \;\; z_P^{(i)}]^{\mathrm{T}}; \tag{5}$$

– $X_P^{(i)}$, $Y_P^{(i)}$, $Z_P^{(i)}$ – the coordinates of the point P of the element i with respect to the fixed reference system O_0XYZ;

– $\{\mathbf{R}_P^{(i)}\}$ – the column matrix

$$\{\mathbf{R}_P^{(i)}\} = [X_P^{(i)} \;\; Y_P^{(i)} \;\; Z_P^{(i)}]^{\mathrm{T}}; \tag{6}$$

– f, g – the equations of the surface or curve;

– $\left[\overline{\mathbf{A}}_i\right]$ – the rotational matrix in the planar case

$$\left[\overline{\mathbf{A}}_i\right] = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix}. \tag{7}$$

3. The case of a point constrained to move on a fixed surface

Let be

$$f(X, Y, Z) = 0 \tag{8}$$

the equation of the fixed surface (Σ) relative to the fixed reference system (Fig. 1).

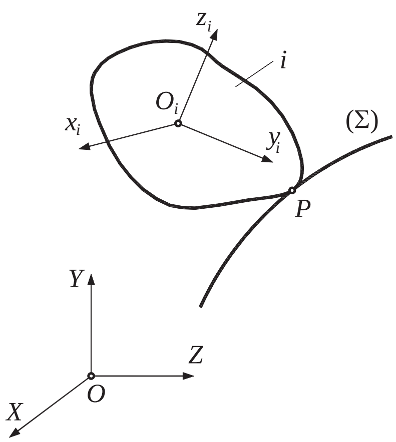

Fig. 1. A point constrained to move on a fixed surface.

The coordinates of the point P that belongs to the element i, constrained to move on the surface (Σ) may be written as

$$\left\{\mathbf{R}_P^{(i)}\right\} = \left\{\mathbf{R}_{O_i}\right\} + [\mathbf{A}]\left\{\mathbf{r}_P^{(i)}\right\} = \begin{bmatrix} X_P^{(i)} \\ Y_P^{(i)} \\ Z_P^{(i)} \end{bmatrix}; \tag{9}$$

it results the constraint function

$$f_1\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = f\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = 0\,. \tag{10}$$

In conclusion, the system looses one degree of freedom.

4. The case of a point constrained to move on a fixed curve

4.1. The general case

Proceeding as in the paragraph 3, one obtains the constraint functions

$$f_1\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = f\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = 0\,, \tag{11}$$

$$f_2\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = g\left(X_P^{(i)}, Y_P^{(i)}, Z_P^{(i)}\right) = 0,$$

where

$$f(X, Y, Z) = 0,\ g(X, Y, Z) = 0 \tag{12}$$

are the equations of the curve in the fixed reference system.

We have now two functions of constraints and, consequently, the system looses two degrees of freedom.

4.2. The planar case

The situation is presented in Fig. 2.

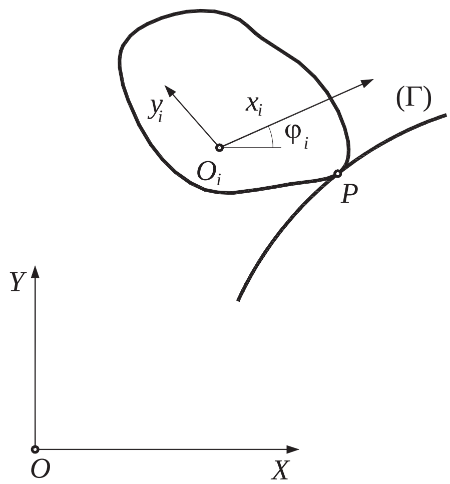

Fig. 2. A point constrained to move on a fixed curve.

Let be

$$f(X, Y) = 0 \tag{13}$$

the equation of the curve (Γ) in the fixed reference frame.

We may write

$$\begin{bmatrix} X_P^{(i)} \\ Y_P^{(i)} \end{bmatrix} = \begin{bmatrix} X_{Oi} \\ Y_{O_i} \end{bmatrix} + [\overline{\mathbf{A}}_i] \begin{bmatrix} x_P^{(i)} \\ y_P^{(i)} \end{bmatrix} = \begin{bmatrix} X_{Oi} \\ Y_{O_i} \end{bmatrix} + \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix} \begin{bmatrix} x_P^{(i)} \\ y_P^{(i)} \end{bmatrix}, \tag{14}$$

wherefrom

$$X_P^{(i)} = X_{O_i} + x_P^{(i)}\cos\varphi_i - y_P^{(i)}\sin\varphi_i,\ Y_P^{(i)} = Y_{O_i} + x_P^{(i)}\sin\varphi_i + y_P^{(i)}\cos\varphi_i; \tag{15}$$

it results the constraint function

$$\begin{gathered} f_1\left(X_P^{(i)}, Y_P^{(i)}\right) = f\left(X_P^{(i)}, Y_P^{(i)}\right) \\ = f\left(X_{O_i} + x_P^{(i)}\cos\varphi_i - y_P^{(i)}\sin\varphi_i, Y_{O_i} + x_P^{(i)}\sin\varphi_i + y_P^{(i)}\cos\varphi_i\right) = 0. \end{gathered} \tag{16}$$

The system looses one degree of freedom.

5. The case of the spherical joint

The common point P of the spherical joint has the coordinates $x_P^{(i)}$, $y_P^{(i)}$, $z_P^{(i)}$ relative to the mobile reference system $O_i x_i y_i z_i$, and the coordinates $x_P^{(i+1)}$, $y_P^{(i+1)}$, $z_P^{(i+1)}$ relative to the mobile reference system $O_{i+1} x_{i+1} y_{i+1} z_{i+1}$ (Fig. 3).

It results

$$\left\{\mathbf{R}_P^{(i)}\right\} = \left\{\mathbf{R}_{O_i}\right\} + \left[\mathbf{A}_i\right]\left\{\mathbf{r}_P^{(i)}\right\}, \tag{17}$$

$$\left\{\mathbf{R}_P^{(i+1)}\right\} = \left\{\mathbf{R}_{O_{i+1}}\right\} + \left[\mathbf{A}_{i+1}\right]\left\{\mathbf{r}_P^{(i+1)}\right\} \tag{18}$$

and from the equality

$$\left\{\mathbf{R}_P^{(i)}\right\} = \left\{\mathbf{R}_P^{(i+1)}\right\} \tag{19}$$

one obtains three functions of constraints, that is, the system consisting in the elements i and $i+1$ looses three degrees of freedom.

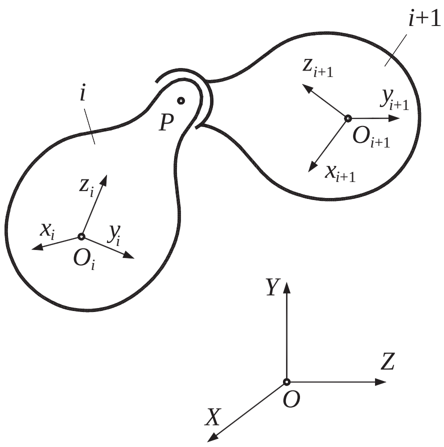

Fig. 3. The spherical joint.

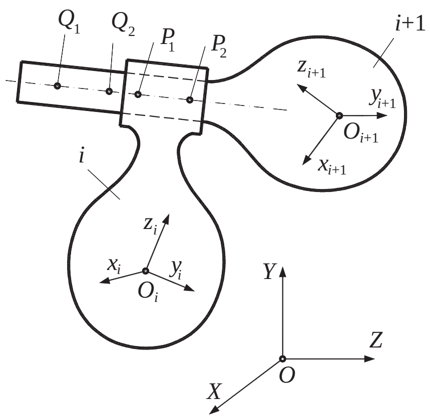

Fig. 4. The cylindrical joint.

6. The cylindrical joint

Let be P_1 and P_2 two points situated on the axis of the cylindrical joint and which are rigidly linked to the element i, and let be Q_1 and Q_2 two other points situated on the same axis and rigidly linked to the element $i+1$ (Fig. 4). The four points can be real or virtual.

At an arbitrary moment of time, one may write the relations

$$\{\mathbf{R}_{P_1}^{(i)}\} = \{\mathbf{R}_{O_i}\} + [\mathbf{A}_i]\{\mathbf{r}_{P_1}^{(i)}\}, \tag{20}$$

$$\{\mathbf{R}_{P_2}^{(i)}\} = \{\mathbf{R}_{O_i}\} + [\mathbf{A}_i]\{\mathbf{r}_{P_2}^{(i)}\}, \tag{21}$$

$$\{\mathbf{R}_{Q_1}^{(i+1)}\} = \{\mathbf{R}_{O_{i+1}}\} + [\mathbf{A}_{i+1}]\{\mathbf{r}_{Q_1}^{(i+1)}\}, \tag{22}$$

$$\{\mathbf{R}_{Q_2}^{(i+1)}\} = \{\mathbf{R}_{O_{i+1}}\} + [\mathbf{A}_{i+1}]\{\mathbf{r}_{Q_2}^{(i+1)}\}. \tag{23}$$

From the previous relations one deduces the coordinates $X_{P_1}^{(i)}$, $Y_{P_1}^{(i)}$, $Z_{P_1}^{(i)}$, $X_{P_2}^{(i)}$, $Y_{P_2}^{(i)}$, $Z_{P_2}^{(i)}$, $X_{Q_1}^{(i+1)}$, $Y_{Q_1}^{(i+1)}$, $Z_{Q_1}^{(i+1)}$, and $X_{Q_2}^{(i+1)}$, $Y_{Q_2}^{(i+1)}$, $Z_{Q_2}^{(i+1)}$ of the points P_1, P_2, Q_1, and Q_2, respectively, relative to the fixed reference system. The points P_1 and P_2 define a straight line, the equations of which with respect to the fixed reference system read

$$f(X, Y, Z) = 0,\ g(X, Y, Z) = 0. \tag{24}$$

The constraint functions require the appurtenance of the points Q_1 and Q_2 to this straight line. One obtains four functions of constraints

$$\begin{aligned} f\left(X_{Q_1}^{(i+1)}, Y_{Q_1}^{(i+1)}, Z_{Q_1}^{(i+1)}\right) = 0,\ g\left(X_{Q_1}^{(i+1)}, Y_{Q_1}^{(i+1)}, Z_{Q_1}^{(i+1)}\right) = 0, \\ f\left(X_{Q_2}^{(i+1)}, Y_{Q_2}^{(i+1)}, Z_{Q_2}^{(i+1)}\right) = 0,\ g\left(X_{Q_2}^{(i+1)}, Y_{Q_2}^{(i+1)}, Z_{Q_2}^{(i+1)}\right) = 0, \end{aligned} \tag{25}$$

the number of the degrees of freedom of the system diminishing by four.

7. The planar revolute joint

One obtains the relations (Fig. 5)

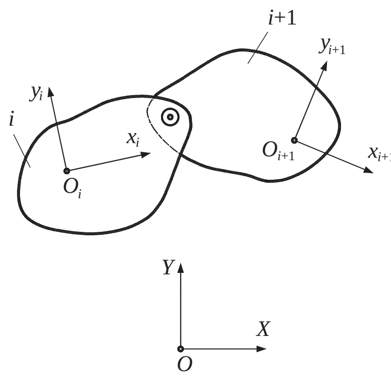

Fig. 5. The planar revolute joint.

$$\left\{\mathbf{R}_P^{(i)}\right\} = \begin{bmatrix} X_P^{(i)} \\ Y_P^{(i)} \end{bmatrix} = \begin{bmatrix} X_{O_i} \\ Y_{O_i} \end{bmatrix} + \left[\overline{\mathbf{A}}_i\right] \begin{bmatrix} x_P^{(i)} \\ y_P^{(i)} \end{bmatrix} = \begin{bmatrix} X_{O_i} \\ Y_{O_i} \end{bmatrix} + \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix} \begin{bmatrix} x_P^{(i)} \\ y_P^{(i)} \end{bmatrix}, \tag{26}$$

$$\begin{aligned} \left\{\mathbf{R}_P^{(i+1)}\right\} &= \begin{bmatrix} X_P^{(i+1)} \\ Y_P^{(i+1)} \end{bmatrix} = \begin{bmatrix} X_{O_{i+1}} \\ Y_{O_{i+1}} \end{bmatrix} + \left[\overline{\mathbf{A}}_{i+1}\right] \begin{bmatrix} x_P^{(i+1)} \\ y_P^{(i+1)} \end{bmatrix} \\ &= \begin{bmatrix} X_{O_{i+1}} \\ Y_{O_{i+1}} \end{bmatrix} + \begin{bmatrix} \cos\varphi_{i+1} & -\sin\varphi_{i+1} \\ \sin\varphi_{i+1} & \cos\varphi_{i+1} \end{bmatrix} \begin{bmatrix} x_P^{(i+1)} \\ y_P^{(i+1)} \end{bmatrix} \end{aligned} \tag{27}$$

and from the condition

$$\left\{\mathbf{R}_P^{(i)}\right\} = \left\{\mathbf{R}_P^{(i+1)}\right\} \tag{28}$$

one deduces two functions of constraints

$$\begin{aligned} f_1\left(X_{O_i}, Y_{O_i}, \varphi_i, X_{O_{i+1}}, Y_{O_{i+1}}, \varphi_{i+1}\right) &= X_{O_i} + x_P^{(i)}\cos\varphi_i - y_P^{(i)}\sin\varphi_i \\ &- X_{O_{i+1}} - x_P^{(i+1)}\cos\varphi_{i+1} + y_P^{(i+1)}\sin\varphi_{i+1} = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} f_2\left(X_{O_i}, Y_{O_i}, \varphi_i, X_{O_{i+1}}, Y_{O_{i+1}}, \varphi_{i+1}\right) &= Y_{O_i} + x_P^{(i)}\sin\varphi_i + y_P^{(i)}\cos\varphi_i \\ &- Y_{O_{i+1}} - x_P^{(i+1)}\sin\varphi_{i+1} - y_P^{(i+1)}\cos\varphi_{i+1} = 0. \end{aligned} \tag{30}$$

In this situation, the system looses two degrees of freedom.

8. The prismatic joint

8.1. The general case

We choose the points P_1, P_2, Q_1 and Q_2 as in the case of the cylindrical joint (paragraph 6). Similarly, one gets the functions of constraints (25) (Fig. 6).

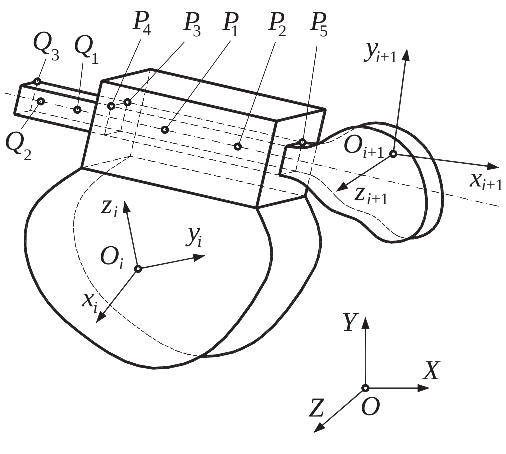

Fig. 6. The spatial prismatic joint

In addition, we choose the points P_3, P_4 and P_5 belonging to the element i, situated on a face of the prism, and the point Q_3, belonging to the element $i+1$ and situated in the plane defined by the points P_3, P_4 and P_5. One may write

$$\left\{\mathbf{R}_{P_{3,4,5}}^{(i)}\right\} = \left\{\mathbf{R}_{O_i}\right\} + \left[\mathbf{A}_i\right]\left\{\mathbf{r}_{P_{3,4,5}}^{(i)}\right\}, \tag{31}$$

$$\left\{\mathbf{R}_{Q_3}^{(i+1)}\right\} = \left\{\mathbf{R}_{O_{i+1}}\right\} + \left[\mathbf{A}_{i+1}\right]\left\{\mathbf{r}_{Q_3}^{(i+1)}\right\}. \tag{32}$$

The condition which we are looking for requires the appurtenance of the point Q_3 to the plane defined by the points P_3, P_4 and P_5, that is,

$$\begin{vmatrix} 1 & X_{P_3}^{(i)} & Y_{P_3}^{(i)} & Z_{P_3}^{(i)} \\ 1 & X_{P_4}^{(i)} & Y_{P_4}^{(i)} & Z_{P_4}^{(i)} \\ 1 & X_{P_5}^{(i)} & Y_{P_5}^{(i)} & Z_{P_5}^{(i)} \\ 1 & X_{Q_3}^{(i+1)} & Y_{Q_3}^{(i+1)} & Z_{Q_3}^{(i+1)} \end{vmatrix} = 0. \tag{33}$$

We have five functions of constraints and, consequently, the system looses five degrees of freedom.

8.2. The planar case

The problem is more difficult in this case (Fig. 7).

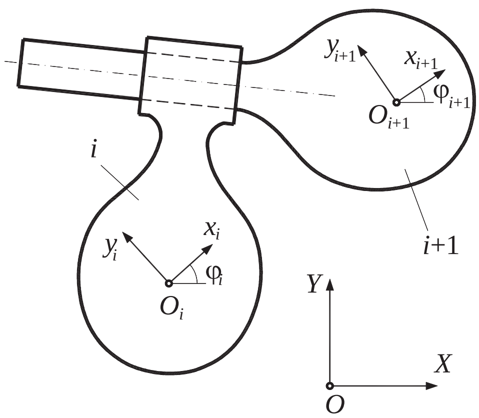

Fig. 7. The planar prismatic joint

The condition which states that the elements i and $i+1$ do not rotate one about other (in plane) leads to the function of constraints

$$f_1 = \varphi_{i+1} - \varphi_i = \text{ct}. \tag{34}$$

The rotation about the axis of the joint can be vanished directly by an obvious function of constraints. It results from the conditions which state that the points O_i and O_{i+1} (if these points do not belong to the rotational axis, otherwise one has to choose other two points) belong to the fixed plane OXY.

9. Example

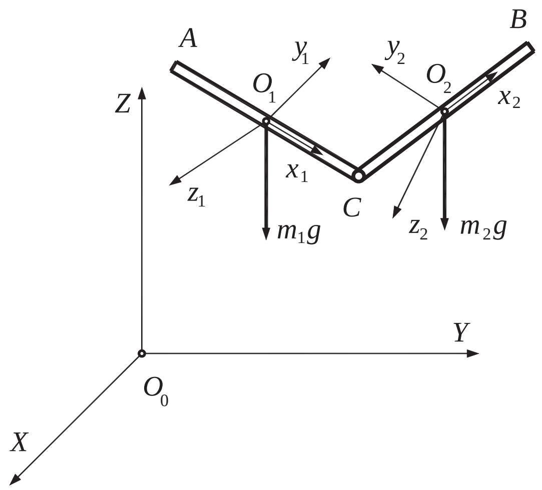

Fig. 8. Example

Let us consider two bars AC and CB of lengths l_1 and l_2, masses m_1 and m_2, respectively, linked one to another by a spherical joint at the point C. The mobile reference frames $O_1x_1y_1z_1$, $O_2x_2y_2z_2$ are principal central systems of inertia for which one knows the values J_{x_1}, J_{y_1}, J_{z_1}, J_{x_2}, J_{y_2}, J_{z_2}. For the fixed reference system O_0XYZ the axis O_0Z is vertical ascendant. Obviously, knowing all these parameters, one may continue the example in order to obtain the matrix equation of motion [12].

Choosing the Bryan angles as rotational parameters for each bar, we have

$$[\boldsymbol{\psi}_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_i & -\sin\psi_i \\ 0 & \sin\psi_i & \cos\psi_i \end{bmatrix}, \ [\boldsymbol{\theta}_i] = \begin{bmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix},$$
$$[\boldsymbol{\varphi}_i] = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ i = \overline{1,2}, \tag{i}$$

$$[\mathbf{A}_i] = [\boldsymbol{\psi}_i][\boldsymbol{\theta}_i][\boldsymbol{\varphi}_i] =$$
$$= \begin{bmatrix} c\theta_i c\varphi_i & -c\theta_i s\varphi_i & s\theta_i \\ s\psi_i s\theta_i c\varphi_i + c\psi_i s\varphi_i & -s\psi_i s\theta_i s\varphi_i + c\psi_i c\varphi_i & -s\psi_i c\theta_i \\ c\psi_i s\theta_i c\varphi_i + s\psi_i s\varphi_i & c\psi_i s\theta_i s\varphi_i + s\psi_i c\varphi_i & c\psi_i c\theta_i \end{bmatrix}, \ i = \overline{1,2}, \tag{ii}$$

We will the consider the following order of the parameters X_{O_1}, Y_{O_1}, Z_{O_1}, ψ_1, θ_1, φ_1, X_{O_2}, Y_{O_2}, Z_{O_2}, ψ_2, θ_2, φ_2.

We have only one constraint given by the appurtenance of the point C to the two bars.

For the bar AC one may write $x_C = \frac{l_1}{2}$, $y_C = 0$, $z_C = 0$, while for the bar BC one has $x_C = -\frac{l_2}{2}$, $y_C = 0$, $z_C = 0$.

From the relations

$$\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \begin{bmatrix} X_{O_1} \\ Y_{O_1} \\ Z_{O_1} \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{bmatrix}, \tag{iii}$$

$$\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \begin{bmatrix} X_{O_2} \\ Y_{O_2} \\ Z_{O_2} \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} -\frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix} \tag{iv}$$

one gets the expressions

$$\begin{gathered} X_{O_1} + \frac{l_1}{2}\cos\theta_1\cos\varphi_1 = X_{O_2} - \frac{l_2}{2}\cos\theta_2\cos\varphi_2, \\ Y_{O_1} + \frac{l_1}{2}(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1) = \\ = Y_{O_2} - \frac{l_2}{2}(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2), \\ Z_{O_1} + \frac{l_1}{2}(-\cos\psi_1\sin\theta_1\cos\varphi_1 + \sin\psi_1\sin\varphi_1) = \\ = Z_{O_2} - \frac{l_2}{2}(-\cos\psi_2\sin\theta_2\cos\varphi_2 + \sin\psi_2\sin\varphi_2), \end{gathered} \tag{v}$$

so that the functions of constraints read

$$\begin{gathered} f_1(X_{O_1}, \ldots, \varphi_2) = X_{O_1} - X_{O_2} + \frac{l_1}{2}\cos\theta_1\cos\varphi_1 + \frac{l_2}{2}\cos\theta_2\cos\varphi_2 = 0, \\ f_2(X_{O_1}, \ldots, \varphi_2) = Y_{O_1} - Y_{O_2} + \frac{l_1}{2}(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1) \\ + \frac{l_2}{2}(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2) = 0, \end{gathered} \tag{vi}$$

$$f_3\left(X_{O_1},...,\varphi_2\right)=Z_{O_1}-Z_{O_2}+\frac{l_1}{2}\left(-\cos\psi_1\sin\theta_1\cos\varphi_1+\sin\psi_1\sin\varphi_1\right)$$
$$+\frac{l_2}{2}\left(-\cos\psi_2\sin\theta_2\cos\varphi_2+\cos\psi_2\sin\varphi_2\right)=0,$$

wherefrom one obtains the components of the matrix of constraints as follows

$$B_{11}=\frac{\partial f_1}{\partial X_{O_1}}=1,\ B_{12}=\frac{\partial f_1}{\partial Y_{O_1}}=0,\ B_{13}=\frac{\partial f_1}{\partial Z_{O_1}}=0,\ B_{14}=\frac{\partial f_1}{\partial\psi_1}=0,$$
$$B_{15}=\frac{\partial f_1}{\partial\theta_1}=-\frac{l_1}{2}\sin\theta_1\cos\varphi_1,\ B_{16}=\frac{\partial f_1}{\partial\varphi_1}=-\frac{l_1}{2}\cos\theta_1\sin\varphi_1,$$
$$B_{17}=\frac{\partial f_1}{\partial X_{O2}}=-1,\ B_{18}=\frac{\partial f_1}{\partial Y_{O2}}=0,\ B_{19}=\frac{\partial f_1}{\partial Z_{O2}}=0,\ B_{110}=\frac{\partial f_1}{\partial\psi_2}=0,\tag{vii}$$
$$B_{111}=\frac{\partial f_1}{\partial\theta_2}=-\frac{l_2}{2}\sin\theta_2\cos\varphi_2,\ B_{112}=\frac{\partial f_1}{\partial\varphi_2}=-\frac{l_2}{2}\cos\theta_2\sin\varphi_2$$

$$B_{21}=\frac{\partial f_2}{\partial X_{O_1}}=0,\ B_{22}=\frac{\partial f_2}{\partial Y_{O_1}}=1,\ B_{23}=\frac{\partial f_2}{\partial Z_{O_1}}=0,$$
$$B_{24}=\frac{\partial f_2}{\partial\psi_1}=\frac{l_1}{2}\left(\cos\psi_1\sin\theta_1-\sin\psi_1\sin\varphi_1\right),$$
$$B_{25}=\frac{\partial f_2}{\partial\theta_1}=\frac{l_1}{2}\sin\psi_1\cos\theta_1\cos\varphi_1,$$
$$B_{26}=\frac{\partial f_2}{\partial\varphi_1}=\frac{l_1}{2}\left(-\sin\psi_1\sin\theta_1\sin\varphi_1+\cos\psi_1\cos\varphi_1\right),\ B_{27}=\frac{\partial f_2}{\partial X_{O2}}=0,$$
$$B_{28}=\frac{\partial f_2}{\partial Y_{O2}}=-1,\ B_{29}=\frac{\partial f_2}{\partial Z_{O2}}=0,\tag{viii}$$
$$B_{210}=\frac{\partial f_2}{\partial\psi_2}=\frac{l_2}{2}\left(\cos\psi_2\sin\theta_2\cos\varphi_2-\sin\psi_2\sin\varphi_2\right),$$
$$B_{211}=\frac{\partial f_2}{\partial\theta_2}=\frac{l_2}{2}\sin\psi_2\cos\theta_2\cos\varphi_2,$$
$$B_{212}=\frac{\partial f_2}{\partial\varphi_2}=\frac{l_2}{2}\left(-\sin\psi_2\sin\theta_2\sin\varphi_2+\cos\psi_2\cos\varphi_2\right),$$

$$B_{31}=\frac{\partial f_3}{\partial X_{O_1}}=0,\ B_{32}=\frac{\partial f_3}{\partial Y_{O_1}}=0,\ B_{33}=\frac{\partial f_3}{\partial Z_{O_1}}=1,\tag{ix}$$
$$B_{34}=\frac{\partial f_3}{\partial\psi_1}=\frac{l_1}{2}\left(\sin\psi_1\sin\theta_1\cos\varphi_1+\cos\psi_1\sin\varphi_1\right),$$

$$B_{35} = \frac{\partial f_2}{\partial \theta_1} = -\frac{l_1}{2}\cos\psi_1 \cos\theta_1 \cos\varphi_1 ,$$

$$B_{36} = \frac{\partial f_3}{\partial \varphi_1} = \frac{l_1}{2}\left(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1\right),\ B_{37} = \frac{\partial f_3}{\partial X_{O2}} = 0 ,$$

$$B_{38} = \frac{\partial f_3}{\partial Y_{O2}} = 0 ,\ B_{39} = \frac{\partial f_3}{\partial Z_{O2}} = -1 ,$$

$$B_{310} = \frac{\partial f_3}{\partial \psi_2} = \frac{l_2}{2}\left(\sin\psi_2 \sin\theta_2 \cos\varphi_2 + \cos\psi_2 \sin\varphi_2\right),$$

$$B_{311} = \frac{\partial f_3}{\partial \theta_2} = -\frac{l_2}{2}\cos\psi_2 \cos\theta_2 \cos\varphi_2 ,$$

$$B_{312} = \frac{\partial f_3}{\partial \varphi_2} = \frac{l_2}{2}\left(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2\right).$$

Let us consider now that at the point C we have a cylindrical joint.

In this situation we have to add two new functions of constraints

$$f_4\left(X_{O_1},\ldots,\varphi_2\right) = \psi_2 - \psi_1 = 0 ,\ f_5\left(X_{O_1},\ldots,\varphi_2\right) = \theta_2 - \theta_1 = 0 , \tag{x}$$

the matrix of constraints having now 5 rows and 12 columns.

The first three rows were calculated previously; the next two rows have the expressions

$$B_{41} = 0 ,\ B_{42} = 0 ,\ B_{43} = 0 ,\ B_{44} = -1 ,\ B_{45} = 0 ,\ B_{46} = 0 ,\ B_{47} = 0 ,$$
$$B_{48} = 0 ,\ B_{49} = 0 ,\ B_{410} = 1 ,\ B_{411} = 0 ,\ B_{412} = 0 , \tag{xi}$$

$$B_{51} = 0 ,\ B_{52} = 0 ,\ B_{53} = 0 ,\ B_{54} = 0 ,\ B_{55} = -1 ,\ B_{56} = 0 ,\ B_{57} = 0 ,$$
$$B_{58} = 0 ,\ B_{59} = 0 ,\ B_{510} = 0 ,\ B_{511} = 1 ,\ B_{512} = 0 . \tag{xii}$$

Let us impose now the supplementary conditions which state that the points A, and B are situated on the spheres of equations

$$g_1(X,Y,Z) = X^2 + Y^2 + Z^2 - R^2 = 0 , \tag{xiii}$$

and

$$g_2(X,Y,Z) = (X - 2R)^2 + Y^2 + Z^2 - R^2 = 0 , \tag{xiv}$$

respectively.

The points A and B have the coordinates $x_A = -\frac{l_1}{2}$, $y_A = 0$, $z_A = 0$, $x_B = \frac{l_2}{2}$, $y_B = 0$, $z_B = 0$; it results

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} = \begin{bmatrix} X_{O_1} \\ Y_{O_1} \\ Z_{O_1} \end{bmatrix} + [\mathbf{A}_1] \begin{bmatrix} -\frac{l_1}{2} \\ 0 \\ 0 \end{bmatrix} , \tag{xv}$$

$$\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = \begin{bmatrix} X_{O_2} \\ Y_{O_2} \\ Z_{O_2} \end{bmatrix} + [\mathbf{A}_2] \begin{bmatrix} \frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix}, \tag{xvi}$$

that is,

$$X_A = X_{O_1} - \frac{l_1}{2} \cos\theta_1 \cos\varphi_1 ,$$
$$Y_A = Y_{O_1} - \frac{l_1}{2} (\sin\psi_1 \sin\theta_1 \cos\varphi_1 + \cos\psi_1 \sin\varphi_1), \tag{xvii}$$
$$Z_A = Z_{O_1} - \frac{l_1}{2} (-\cos\psi_1 \sin\theta_1 \cos\varphi_1 + \sin\psi_1 \sin\varphi_1),$$

$$X_B = X_{O_2} + \frac{l_2}{2} \cos\theta_2 \cos\varphi_2 ,$$
$$Y_B = Y_{O_2} + \frac{l_2}{2} (\sin\psi_2 \sin\theta_2 \cos\varphi_2 + \cos\psi_2 \sin\varphi_2), \tag{xviii}$$
$$Z_B = Z_{O_2} + \frac{l_2}{2} (-\cos\psi_2 \sin\theta_2 \cos\varphi_2 + \sin\psi_2 \sin\varphi_2).$$

One gets the functions of constraints

$$f_4(X_{O_1}, \ldots, \varphi_2) = \left(X_{O_1} - \frac{l_1}{2} \cos\theta_1 \cos\varphi_1 \right)^2 +$$
$$+ \left[Y_{O_1} - \frac{l_1}{2} (\sin\psi_1 \sin\theta_1 \cos\varphi_1 + \cos\psi_1 \sin\varphi_1) \right]^2 + \tag{xix}$$
$$+ \left[Z_{O_1} - \frac{l_1}{2} (-\cos\psi_1 \sin\theta_1 \cos\varphi_1 + \sin\psi_1 \sin\varphi_1) \right]^2 - R^2 = 0,$$

$$f_5(X_{O_1}, \ldots, \varphi_2) = \left(X_{O_2} - 2R + \frac{l_2}{2} \cos\theta_2 \cos\varphi_2 \right)^2 +$$
$$+ \left[Y_{O_2} + \frac{l_2}{2} (\sin\psi_2 \sin\theta_2 \cos\varphi_2 + \cos\psi_2 \sin\varphi_2) \right]^2 + \tag{xx}$$
$$+ \left[Z_{O_2} + \frac{l_2}{2} (-\cos\psi_2 \sin\theta_2 \cos\varphi_2 + \sin\psi_2 \sin\varphi_2) \right]^2 - R^2 = 0.$$

The matrix of constraints has now 5 rows and 12 columns.

Let us impose the condition which states that the points A and B are situated in the plane $Z = 0$, while at the point C we have a cylindrical joint.

We may successively write (the conditions for the appurtenance of the points A and B to the plane $Z = 0$; the rest of the functions of constraints were previously determined):

$$\begin{bmatrix} X_A \\ Y_A \\ 0 \end{bmatrix} = \begin{bmatrix} X_{O_1} \\ Y_{O_1} \\ Z_{O_1} \end{bmatrix} + [\mathbf{A}_1] \begin{bmatrix} -\frac{l_1}{2} \\ 0 \\ 0 \end{bmatrix}, \tag{xxi}$$

$$Z_{O_1} - \frac{l_1}{2}\left(-\cos\psi_1 \sin\theta_1 \cos\varphi_1 + \sin\psi_1 \sin\varphi_1\right) = 0. \tag{xxii}$$

$$\begin{bmatrix} X_B \\ Y_B \\ 0 \end{bmatrix} = \begin{bmatrix} X_{O_2} \\ Y_{O_2} \\ Z_{O_2} \end{bmatrix} + [\mathbf{A}_2] \begin{bmatrix} \frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix}, \tag{xxiii}$$

$$Z_{O_2} + \frac{l_2}{2}\left(-\cos\psi_2 \sin\theta_2 \cos\varphi_2 + \sin\psi_2 \sin\varphi_2\right) = 0. \tag{xxiv}$$

The matrix of constraints has now 7 rows and 12 columns.

10. Conclusions

This paper presents a multibody approach for the determination of the functions of constraints and, consequently, the matrix of constraints in the case of a general mechanism. The functions of constraints are determined in each particular case. An example shows the application of the theory.

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