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Critical stresses, deterioration and failure criteria for cracked structures

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Abstract: Starting from the concept of deterioration, using the principle of critical energy and considering the nonlinear, power law, behavior of the sample, or of the mechanical structure, there were proposed relations for critical stresses in the case of: – uniaxial static loading with normal stress or with shear stress, or simultaneously with normal and shear stress; – uniaxial cyclic loading with one block of normal stress or one block of shear stress, as well as simultaneous with a block of normal stress and one block of shear stress, taking into account the deterioration due to crack. Failure criteria for cracked mechanical structures, uniaxial statically or cyclically loaded, have been developed. The critical stresses calculated for cracked samples compared with experimental results reported in literature, have shown good agreement between them.

Keywords:, deterioration, critical stress, static and fatigue loading, cracked structure, failure criteria, principle of critical energy.

1. Introduction

Deteriorations means the irreversible reduction of the capacity of a material or structure from a certain point of view: the reduction of the capacity of resistance, of deformation, of heat conductivity etc. If through deterioration the structure can't be used, that means the structure was damaged and becomes unsafe.

The origin of deterioration or of damage in mechanical structures (pressure equipment, gas turbines, ships, aircraft, locomotives, bridges, etc.) is often to be found in the flaws "impressed" during their manufacturing (casting, stamping, forging, welding, riveting etc.) and / or the cracks "born" during use, particularly as

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a result of overloading. Some parts of mechanical structures feature micropores which subsequently generate small cracks during loading. In certain circumstances, these cracks propagate. If not detected through periodical examination and repaired, the cracks propagate down to failure.

The requirements concerning the safety of mechanical structures need us to consider cracks / flaws, both in their design stage and their monitoring during operation. The influence of cracks on mechanical structures has been studied from different points of view, namely: - the influence of axial cracks upon the rupture pressure of a cylindrical pressure vessel [1]; the influence of sloping cracks from a autofretaged cylindrical body upon rupture pressure [2]; the influence of cracks caused by corrosion upon the critical circumferential stress and plastic instability in pipes [3-5]; the influence of a surface crack upon the plate, loaded with force and bending moment, on the limit state [6]; the influence of the circumferential semi-elliptical crack at the outer surface of a cylindrical body under cyclic axial load on fatigue fracture toughness [7]; the influence of circumferential cracks at the outer surface of a thick-walled pipe under rotary bending upon fatigue fracture toughness [8]; the influence of axial cracks on the outer surface of the pipes undergoing pulsating pressure upon the stress intensity factor [9]; the influence of fatigue cracks on the critical stress range [10, 11] etc.

For example, the investigations concerning the bursting pressure of the autofrettaged cylinders with inclined cracks have shown that [2]: – external axial cracks are critical and reduce considerably the bursting pressure (more than 30%); – external circumferential cracks have not considerable effects on reducing burst pressure (less than 6%); – for crack with same length, the deeper crack is critical; – crack growth always starts from initial crack, but immediately transforms to the axial crack, because hoop stresses are the largest principal stresses. Defects or surface cracks on outer surface, due to stress concentrations reduce the pressure capacity of a vessel.

On the other side the effect of damage on the limit loads of the cracked cylinders and thick- walled pipes [12; 13] and the influence of flaws and defects on the fatigue strength have been studied [14 ÷ 17].

The crack influence on the structure mixed-mode loaded were analised with the fracture mechanics concepts [18;19]. The concept of deterioration has been used too [20-26].

In some industries, such as the aerospace industry, the method of "damage tolerance design" has replaced the previously used methods, namely the „safe life method" and the "fail safe method".

Currently, it is acknowledged that cracks do exist and they may evolve during loading up to a maximum limit. The existence of cracks raises the following problems: *a* – is a crack, whose initial sizes are a_i and $2c_i$, dangerous or not? In other words, is there a risk of crack propagation under the specific loading circumstances? *b* - if crack propagation qualifies, how long will it take or how many loading cycles are needed before or after each load cycle for the crack to reach the critical sizes, a_{cr} and $2c_{cr}$?, *c* - how are we to calculate the deterioration,

$D(a;c)$, produced by a crack whose characteristic size is *a* and 2*c* and how we may use it; *d* - how are we to take into account the crack resulting deterioration in design or in the calculations of strength or stiffness?
The answers to the first two problems, *a* and *b* are to be found in Fracture Mechanics, the other two, *c* and *d*, will be treated in this paper.

2. The dependence of critical stresses on the deterioration due to cracks

Kachanov [22] introduced the deterioration parameter *D*, a nondimensional variable which has the following properties: $D = 0$, for the undamaged material and $D = 1$, for the fully damaged of the structure material (failure, excessive deformation etc…). Generally $D \in (0;1)$.
An extended analysis of the deterioration case is, for example, in references [23;24].
One considers the general case of the nonlinear, power law, behavior of the material structure, *statically loaded* under normal, σ, and shear, τ, stresses,

$$\left.\begin{aligned} \sigma &= M_\sigma \cdot \varepsilon^k; \\ \tau &= M_\tau \cdot \gamma^{k1}, \end{aligned}\right\} \quad (1)$$

where ε is strain, γ – shear strain, and M_σ, M_τ, k and k_1 – constants of the material.
The material behavior if *fatigue loaded* is given by the Basquin's relation [27], used by the ASME code, Section VIII, and in the EN 13445,

$$\sigma_a^m \cdot N = A, \quad (2)$$

where σ_a is the normal stress amplitude; *N* is the number of cycles up to failure (fatigue life); A, m – material constant.
The deterioration *D*(*a*;*c*) depends on the material behavior through exponent $\alpha = 1/k$; its value is influenced by the rate of load application (static, fast or shock) [28 ÷ 30].

2.1. Mechanical structures monotonic loaded; critical stresses and failure criteria

In this paper as to solve the proposed problems one resort to the concept of energy, namely to the principle of critical energy [28;31] and to a critical energy approach. This is because the use of critical parameters based on the strain energy have many advantages with respect to stress based criteria, such as it results from several papers discussing this issue [32-34], or showing the applicability of such criterion [35-38].
The critical state at the crack tip is obtained when the condition resulted from principle of critical energy [19; 28],

$$P_T = P_{cr}(t) \quad (3)$$

is fulfilled, where $P_T = \sum_i P_i$, is the total participation of the specific energies [28; 29] and $P_{cr}(t)$ is the critical specific energy participation at the time *t*.

For an undeteriorated sample $P_{cr} = P_{cr}(0)$, where due to scattering of the material mechanical characteristics, $P_{cr}(0)$ also features a statistical distribution, that is $P_{cr}(0) \in [P_{cr,\min}(0); P_{cr,\max}(0)]$, where $P_{cr,\max}(0) \le 1$. Neglecting the residual stresses, generally [30],

$$P_{cr}(t) = P_{cr}(0) - D_T(t), \tag{4}$$

where $D_T(t) = \sum_i D_i(t)$ is the total deterioration.

When the critical state is obtained (failure) the deterioration becomes damage. In Table 1 were listed some relationships of deterioration due to different causes.

Because, by *structure design* the mechanical characteristics are assumed to be deterministic value $P_{cr}(0) = 1$.

Table 1. Some relationships for the deterioration calculation

Nr.	**Cause of deterioration**	**Relationship of deterioration**	**Observations**
1.	Cyclic loading [30;48]	$D(\sigma_a; N) = \left(\frac{n}{N}\right)^{\frac{\alpha+1}{m}}$	*m* – the exponent of the Basquin's law (2); $\alpha = 1/k$.
2.	Vibration [23]	$D(\omega) = \left(\frac{\omega}{\omega_{cr}}\right)^2$	ω – pulsation; ω_{cr} – critical pulsation value.
3.	Uniform corrosion [23]	$D(t_{cs}) = \left(\frac{t}{t_{cs}}\right)^c$	*t* – effective corrosion time; t_{cs} – time until complete corrosion; *c* – constant
4.	Creep [30]	$D(t_c) = \left(\frac{t}{t_c}\right)^{\frac{\alpha+1}{f}}$	*t* – time of stressing in the creep condition; t_c – critical value of *t*; $\alpha = 1/k$; *f* – becomes from the slop of the creep rupture curve.
5.	Crack [25;48]	$D(a;c) = D(c) \cdot [1 + D(a)]$; $D(a) = \left(\frac{a}{a_{cr}}\right)^{\frac{\alpha+1}{2}}$; $D(c) = \left(\frac{c}{2c_{cr}}\right)^{\frac{\alpha+1}{2}}$.	$\alpha = 1/k$

a. Monotonic uniaxial loading

By using the principle of critical energy [28] with $P_T = P(\sigma)$ where $P(\sigma) = (\sigma/\sigma_{cr})^{\alpha+1}$ with $\alpha = 1/k$, it has been obtained,

$$\sigma_{cr}(a;c) = \sigma_{cr} \cdot [P_{cr}(0) - D_\sigma(a;c)]^{\frac{1}{\alpha+1}}, \quad (5)$$

where $\sigma_{cr}(a;c)$ is the critical normal stress of the structure with a crack, uniaxial loaded; σ_{cr} is the critical normal stress of the crackless material and $D_\sigma(a;c)$ is the deterioration due to crack in the case of loading by normal stresses. If the critical characteristics of the material are deterministic values ($P_{cr}(0) = 1$),

$$\sigma_{cr}(a;c) = \sigma_{cr} \cdot [1 - D(a;c)]^{\frac{1}{\alpha+1}}. \quad (6)$$

If the crack sizes a and $2c$ are time dependent, then the deterioration is time dependent, such as,

$$D_T(t) = D[a(t);c(t)].$$

For a sample with cracks statically loaded, the ultimate stress $\sigma_u(a;\ c) = \sigma_{cr}(a;c)$ decreases with the crack increase. In this case $\sigma_{cr} = \sigma_u$ is the ultimate stress of the crackless mechanical structure.

If it is imperative that the material should not exceed the yield stress σ_y, then $\sigma_{cr} = \sigma_y$ and $\alpha = 1$ (for $\sigma \le \sigma_y$ the material behavior is linear - elastic). But in case one allows for the yield stress to be exceeded, then $\sigma_{cr} = \sigma_u$ (ultimate stress).

b. Monotonic shear stress loading

If the mechanical characteristics features statistical distribution, the critical shear stress of the cracked structure is,

$$\tau_{cr}(a;c) = \tau_{cr} \cdot [P_{cr}(0) - D_\tau(a;c)]^{\frac{1}{\alpha_1+1}}, \quad (7)$$

where τ_{cr} is the critical shear stress of the crackless sample; $\alpha_1 = 1/k_1$ and $D_\tau(a;c)$ is calculated by using the size of the crack, characteristic of the corresponding loading τ. If the critical characteristics of the material are deterministic values $(P_{cr}(0) = 1)$,

$$\tau_{cr}(a;c) = \tau_{cr} \cdot [1 - D(a;c)]^{\frac{1}{\alpha_1+1}}. \quad (8)$$

If the sizes a and $2c$ are time dependent, then $D_\tau(a;c) \equiv D_\tau[a(t);c(t)]$ is time dependent, too.

Geometric discontinuities such as holes, grooves, notches, abrupt changes in cross section etc., cause a local increase in stress and a decrease of the local critical stress.

On the basis of experimental data for $\sigma_{cr}(a;c)$ or $\tau_{cr}(a;c)$ the deterioration obtains – for example – from the eqs. (5) and (7):

$$D_\sigma(a;c)=\left(1-\frac{\sigma_{cr}(a;c)}{\sigma_{cr}}\right)^{\alpha+1} \text{ and } D_\tau(a;c)=\left(1-\frac{\tau_{cr}(a;c)}{\tau_{cr}}\right)^{\alpha_1+1}. \qquad (9)$$

c. Monotonic loading simultaneously with a normal and a shear stress of a cracked sample

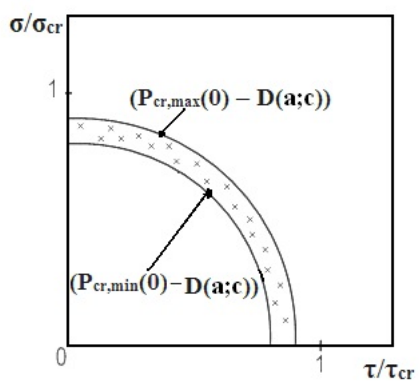


Fig. 1 Superposition of static loadings, σ and τ, according to relation (10) in the case of a cracked sample.

According to the principle of critical energy [28;31] the total participation P_T is the sum of the participations of specific energies $(P(\sigma)$ and $P(\tau))$, corresponding to σ and τ,

$$P_T = P(\sigma)+P(\tau).$$

Results following *failure criterion*,

$$\left(\frac{\sigma}{\sigma_{cr}}\right)^{\alpha+1}+\left(\frac{\tau}{\tau_{cr}}\right)^{\alpha_1+1}=P_{cr}(0)-D(a;c), \qquad (10)$$

where $D(a;c)$ is the deterioration due to the crack. Relation (10) represents a curve that is the geometrical locus of the loading points with σ/σ_{cr} and τ/τ_{cr} when rupture occurs (Fig. 1). If $P_{cr}(0)\in\lfloor P_{cr,\min}(0); P_{cr,\max}(0)\rfloor$ then there are obtained two curves, wherein the experiments points are introduced (Fig. 1).

If the critical stresses depend on the damage caused by the presence of the crack ((5) and (7)), the *failure criterion* (10) may be written as,

$$\left(\frac{\sigma}{\sigma_{cr}(a;c)}\right)^{\alpha+1}+\left(\frac{\tau}{\tau_{cr}(a;c)}\right)^{\alpha_1+1}=1. \qquad (11)$$

d. Monotonic loading simultaneously with blocks of normal stresses and shear stresses

According to the principle of critical energy [28;31], was obtained the following general failure c*riterion* for cracked structures monotonic loaded, with normal stresses σ_i and shear stresses τ_j,

$$\sum_i \left(\frac{\sigma_i}{\sigma_{i,cr}} \right)^{\alpha+1} \cdot \delta_{\sigma_i} + \sum_j \left(\frac{\tau_j}{\tau_{j,cr}} \right)^{\alpha_1+1} \cdot \delta_{\tau_j} = 1 - D_T(t), \qquad (12)$$

which may be also written as the following *failure criterion*,

$$\sum_i \left(\frac{\sigma_i}{\sigma_{i,cr}(a;c)} \right)^{\alpha+1} \cdot \delta_{\sigma_i} + \sum_j \left(\frac{\tau_j}{\tau_{j,cr}(a;c)} \right)^{\alpha_1+1} \cdot \delta_{\tau_j} = 1, \qquad (13)$$

where the critical stresses, as in the case of relationships (11) given by relations (5) and (7). $\delta_{\sigma_i} = 1$ if the normal stress acts in the direction of the process; $\delta_{\sigma_i} = -1$ if it opposes the evolution of the process. δ_{τ_j} has the same interpretation as δ_{σ_i}.

In conclusion, one has found that the algebraic sum of specific energy participations (13) with respect to the critical state of the damaged sample, is equal to 1.0.

2.2. Mechanical structures with cracks, cyclically loaded; critical stresses and failure criteria

a. Under *cyclic loading with only normal stresses* ranging between σ_{min} and σ_{max}:

$$\left.\begin{aligned} &\text{- stress amplitude is } \sigma_a = 0.5(\sigma_{max} - \sigma_{min}) \\ &\text{- mean stress is } \sigma_m = 0.5(\sigma_{max} + \sigma_{min}). \end{aligned}\right\} \qquad (14)$$

The total participation of the specific energies according to the principle of critical energy [28] is

$$P_T(\sigma) = P(\sigma_a) + P(\sigma_m),$$

where from with one obtains the following *failure criterion* for the sample,

$$\left(\frac{\sigma_a}{\sigma_{a,cr}} \right)^{\alpha+1} + \left(\frac{\sigma_m}{\sigma_{m,cr}} \right)^{\alpha+1} \cdot \delta_{\sigma_m} = P_{cr}(0) - D_\sigma(a;c), \qquad (15)$$

where $\sigma_{a,cr} = \sigma_{-1}(N)$ is the fatigue normal strength for the *sample* after fully reversed stress under *N* cycles; $\sigma_{m,cr} = \sigma_y$ (yield stress) or σ_u (ultimate stress), depending on whether fracture occurs when $\sigma_{max} \le \sigma_y$ or $\sigma_y < \sigma_{max} \le \sigma_u$; $\delta_{\sigma_m} = 1$ if $\sigma_m > 0$ (elongation) and $\delta_{\sigma_m} = -1$ if $\sigma_m < 0$ (compression). The deterioration $D_\sigma(a;c)$ is time dependent, because $a = a(n)$ and $c = c(n)$ depend on the number of cycles *n*. In the case of deterministic mechanical characteristics $P_{cr}(0) = 1$.

From relation (15) one may write, for *sample cyclically loaded*, the following *failure criterion*,

$$\left(\frac{\sigma_a}{\sigma_{-1}(a;c;n)}\right)^{\alpha+1}+\left(\frac{\sigma_m}{\sigma_{m,cr}(a;c)}\right)^{\alpha+1}\cdot\delta_{\sigma_m}=1\,, \tag{16}$$

where the critical stresses depends on the crack – perpendicular to the normal stress – and on the number of loading cycles (n), as follows:

$$\left.\begin{aligned}\sigma_{-1}(a;c;n)&=\sigma_{-1}(n)\cdot\left[P_{cr}(0)-D_\sigma(a;c)\right]^{\frac{1}{\alpha+1}};\\ \sigma_{m,cr}(a;c)&=\sigma_{m,cr}\cdot\left[P_{cr}(0)-D_\sigma(a;c)\right]^{\frac{1}{\alpha+1}},\end{aligned}\right\} \tag{17}$$

where $\sigma_{-1}(n)$ is the fatigue limit at *n* cycles, of the a undeteriorated sample; $\sigma_{m,cr}$ is the critical value of σ_m for the undeteriorated sample.

In consequence, because $a=a(n)$ and $c=c(n)$, the critical stress of the cracked sample cyclically loaded, depends upon the time $t=n\cdot T_\sigma$, where T_σ is the period of stress cycle.

b. Under *cyclic loading with only shear stresses* one obtains, by a similar procedure as for normal stresses, *failure criterion*, replacing σ by τ. For example, the critical fatigue shear strength of a cracked sample,

$$\left.\begin{aligned}\tau_{-1}(a,c,n)&=\tau_{-1}(n)\cdot\left[P_{cr}(0)-D_\tau(a;c)\right]^{\frac{1}{\alpha_1+1}};\\ \tau_{m,cr}(a,c)&=\tau_{m,cr}\cdot\left[P_{cr}(0)-D_\tau(a;c)\right]^{\frac{1}{\alpha_1+1}}.\end{aligned}\right\} \tag{18}$$

3. Comments

a. Loading with a cyclic normal stress

In the paper [11] of Atzori and Lazzarin the fatigue strength of a sample ($\sigma_{-1}(c;N)$) decreases with crack length (or depth) increase, starting with $c = c_0$ (Fig. 2). For short crack, $c < c_0$, the strength depends only on the number of cycles, $\sigma_{-1}(n)$.

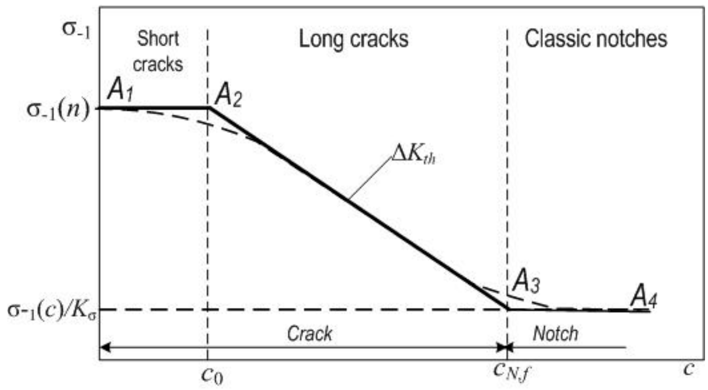


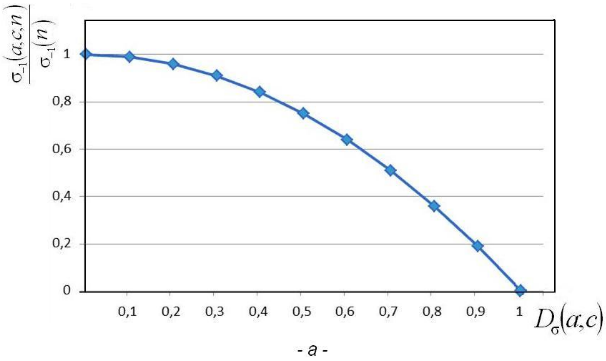
Fig. 2. Fatigue strength versus crack length (or depth) for a sample cyclically loaded (----- real trend) [11].

Atzori and Lazzarin [11] have extended to notches the Kitagawa - Takahashi diagram [39] valid for cracks, with the aim to create a common tool in the analysis of small cracks, long cracks, crack-like notches and common notches [11; 17]. Defect sensitivity and notch sensitivity were seen as two sides of the same medal [11].

Starting from Atzori and Lazzarin observation [11], one can say that a crack with $c > c_{N,f}$ (Fig. 2) behaves like a classic notch for which the fatigue strength is $\sigma_{-1}(n)/K_\sigma$, where K_σ is stress concentration factor. If $\sigma_{-1}(n) \le \sigma_y$, then K_σ is the elastic stress concentration factor. Over the crack size $c_{N,f}$ one may accept that the crack behavior is like a notch. In the Atzori and Lazzarin diagram the curve A_2A_3 depends on the range of stress intensity factor threshold, ΔK_{th}.

On the other side in the paper of Maierhofer et al. [40] the fatigue strength is correlated with a so called total length of the crack.

In our proposal (Fig. 3, *a*) the curve A_2A_3 (Fig. 2) is described by eq. (17); it depends on fatigue stress, namely on deterioration due to crack $D_\sigma(a;c)$ and *is a unique curve for cracks, as well as for notches*. This differs from the Kitagawa-Takahashi diagram [11;39] and from the extended Atzori and Lazzarin diagram [11], where A_2A_3 depends on range of threshold of stress intensity factor, ΔK_{th}.



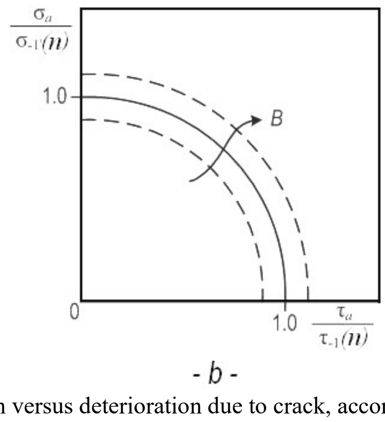


Fig. 3. *a* - Fatigue strength versus deterioration due to crack, according to the first eq. (17) for $P_{cr}(0) = 1$ and a given value of *k*; *b* - diagram of the superposition of cyclic loading with a normal stress and a shear stress, according to relation (19).

b. Loading with cyclic normal stress and shear stress.

When a cyclic loading with normal stress is superposed on the cyclic loading with shear stress, in the same phase, the total participation of the specific energy equals the sum of total participations corresponding to normal stress $P_T(\sigma)$ and shear stress $P_T(\tau)$; it results from Eq. 12, with $i = j = 1$. The following failure criterion for sample is obtained,

$$\left(\frac{\sigma_a}{\sigma_{-1}(n)}\right)^{\alpha+1} + \left(\frac{\tau_a}{\tau_{-1}(n)}\right)^{\alpha_1+1} = B\,, \qquad (19)$$

where one noted,

$$B = P_{cr}(0) - \left(\frac{\sigma_m}{\sigma_{m,cr}}\right)^{\alpha+1} \cdot \delta_{\sigma_m} - \left(\frac{\tau_m}{\tau_{m,cr}}\right)^{\alpha_1+1} \cdot \delta_{\tau_m} - D(a;c),$$

with $\tau_a = 0.5(\tau_{max} - \tau_{min})$ – the shear stress amplitude and $\tau_m = 0.5(\tau_{max} + \tau_{min})$ – the mean shear stress.

Relation (19) describes a curve that represents the geometrical locus of the cyclic loading points in the diagramme $\sigma_a/\sigma_{-1}(n) - \tau_a/\tau_{-1}(n)$ where fracture occurs (Fig. 3, *b*). In case the mechanical characteristics are deterministic quantities, $P_{cr}(0) = 1$.

In the case of linear-elastic behavior $k = 1$ and $\alpha = 1$, such as Eq. (19) describes a quarter of a circle of radius *B*.

4. Comparisons

Several relationships based on the concept of deterioration were reported in literature. These are particular cases of the relationships proposed in our paper.

Based on the concept introduced by Kachanov [22] the *effective stress* at the crack tip, σ_{ef}, is related to deterioration D and to the applied stress, σ, with the relationship $\sigma_{ef} = \sigma/(1-D)$.

Xue and Wierzbicki [40] proposed a relationship of the form $\sigma_{ef} = \sigma/(1-D^{\beta})$, where β is a material constant that is obtained by comparing the relationship with the experimental data. If the material has a linear behavior $\beta = 1$, and Kachanov's Eq. is obtained.

Similarly, the effective stress variation is correlated with the variation in the applied stress, $\Delta\sigma$, namely $\Delta\sigma_{ef} = \Delta\sigma/(1 - D)$.

These relationships are particular cases of relationship (5) where σ_{cr} plays the part of σ_{ef} (or $\Delta\sigma_{ef}$), and $\sigma_{cr}(a;c)$ corresponds to σ (or $\Delta\sigma$).

Gong et al. [41] show that the irreversible accumulated damage $D(n)$ after *n* cycles conducts to progressive reduction from initial values of the cohesive energy, *G*, cohesive stiffness, *k*, and the cohesive strength, σ_u,

$$\left.\begin{array}{l} G(n+1)=G_{Ic}\cdot[1-D(n)]; \\ k(n+1)=k_0\cdot[1-D(n)]\ ; \\ \sigma_u(n+1)=\sigma_u\cdot[1-D(n)]\ , \end{array}\right\} \quad (20)$$

where G_{Ic}, k_0, σ_u are the initial critical values; G_{Ic} is the interface fracture energy; $G(n+1)$, $k(n+1)$ and $\sigma(n+1)$ are the updatet ones after *n* cycles.

On the other hand by coupling the constitutive law with the damage accumulation results and the correlations established by Schwalbe [42 - 45] were obtained [46],

$$\left.\begin{array}{l} \sigma_{cr}(a)=\sigma_{cr}\cdot\left[P_{cr}(0)-\left(K_{\mathrm{I}}/K_{\mathrm{Ic}}\right)^{\alpha+1}\right]^{\frac{1}{\alpha+1}} \\ \text{and} \\ \sigma_{cr}(a)=\sigma_{cr}\cdot\left[P_{cr}(0)-\left(\delta_{\mathrm{I}}/\delta_{\mathrm{Ic}}\right)^{\alpha+1}\right]^{\frac{1}{\alpha+1}}, \end{array}\right\} \quad (21)$$

where K_I is the stress intensity factor for fracture mode I; K_{Ic} fracture toughness or the critical value of K_{I}; $\delta_{\mathrm{I}}; \delta_{\mathrm{Ic}}$ is the crack tip opening displacement and its critical value, respectively, in the case of fracture mode I (opening mode). Eqs. (21) were verified by experimental data in the case $\alpha=1$ [46].

The value of measured plastic Poisson's ratio ν_p and its initial value ν_{p0} have been correlated using a damage parameter *D* [47],

$$\nu_p(D)=\nu_{p0}\cdot(1-D). \quad (22)$$

Equations (20) - (22) show that some of the mechanical properties of the material decrease with the increasing damage caused by cyclic loading (20), the existence of a crack (21) or plastic state deformation (22). The relationships (20) and (22) do not consider the particular behavior of the material under load like in Eqs. (21) and like the way this issue was followed through in the paper [48-53] and in our paper.

5. Experimental validation

To validate the obtained relations with existing experimental results two examples have been used [54 - 57].

a. The diagram in figure 4 shows the dependence of the fully reversed shear strength $\tau_{-1}(n)$ and number of load cycles to failure *n* (Wöhler diagram) for smooth tubular specimens (1) as well as with notch depth a=0.3mm (2) and 0.5mm (3), respectively [56].

Steel tube test pieces with deep cracks a along the generatrix of the external surface have undergone fully reversed loading with the amplitude of shear stress.

The resistance to failure at $N = 10^6$ cycles, calculated with first equation (18) and determined experimentally (Fig. 4), respectively, have been listed in Table 2. Since $\tau_a < \tau_y$, the material behavior is linear - elastic and $\alpha_1 = 1$. The loading being fully reversed $\tau_m = 0$. With $P_{cr}(0) = 0.95$ the first relation (18) becomes:

$$\tau_{-1}(a,n) = \tau_{-1}(n) \cdot [0.95 - D_\tau(a)]^{0.5}, \qquad (23)$$

where $D_\tau(a) = a/a_{cr}$ with $a_{cr} = 13$ mm (thickness of the specimen tube wall).

By comparing the data obtained, one gets a deviation of the shear stress calculated vresus the experimental one of about 19.5% for $a = 0.3$ mm and of -3.14% for $a = 0.5$ mm. Given the relatively large scatter of the mechanical characteristics in fatigue experiments, the theoretical results may be considered satisfactory.

Some other experiments on the influence of flaws on fatigue strength show the decrease of fatigue limit in bending and torsion with initial flaw size [15].

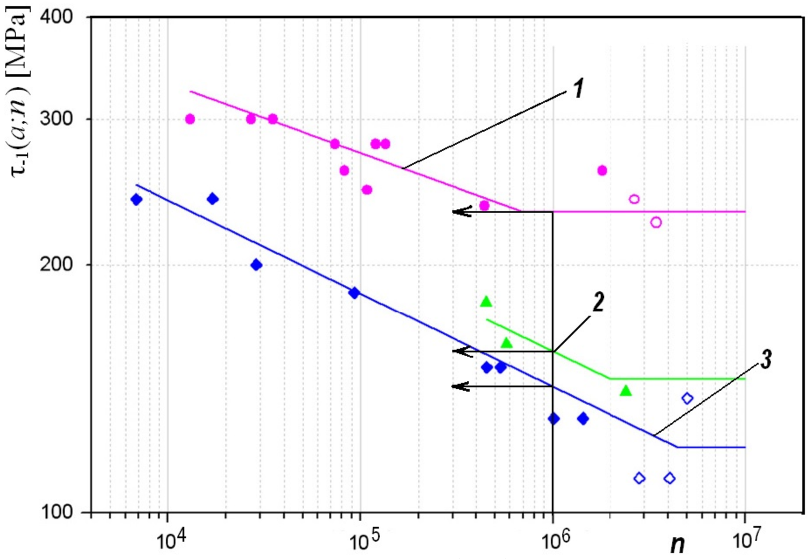


Fig. 4. Diagrame $\tau_{-1}(a;n) - n$

(experimental points from Beretta S., Cerrini A., Desimone H. [56]):
● - smooth specimen (1); ▲ - specimen with notch depth 0.3mm (2);
◆ - specimen with notch depth 0.5mm (3).

Table 2. Experimental shear stress values τ_{-1} (a, n) and calculated one with the relationship (23).

	Unnotched specimen, $a = 0$	Notched specimen			
		$a = 0.3$ mm		$a = 0.5$ mm	
		Experimental [56]	Calculated	Experimental [56]	Calculated
$D_\tau(a)$	0	–	$0.3/1.3 = 0.23077$	–	$0.5/1.3 = 0.3846$
$\tau_{-1}(n)$, MPa	233	–	–	–	–
$\tau_{-1}(a,n)$, MPa	–	150	179.23	136	131.73

b. Starting from the experiments with ceramic plates [57] with one crack located at different sloping angles, α, and crack length 2*c* (Fig. 5, *a*) under uniaxial compression loading values of the compressive strength σ_c, at different values of crack length (Fig. 5, *b*), as well as amount of damage there have been obtained.

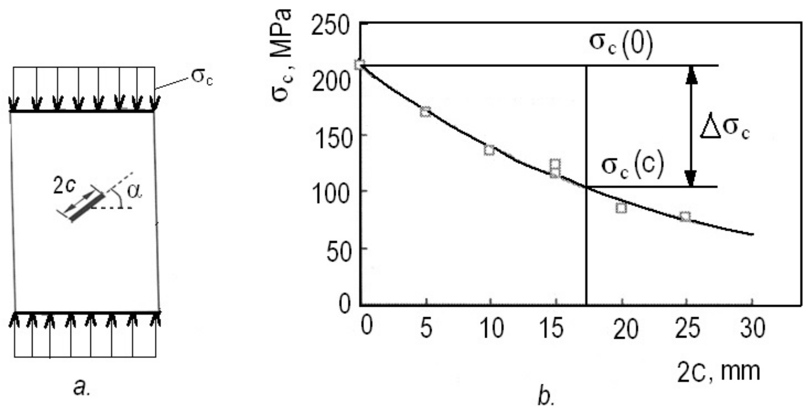


Fig. 5, *a* – The crack geometric parameters; *b* – uniaxial compressive strength variation depending on crack length, 2*c* (experimental points from Sheng – Qi Yang, Hong – Wen Jing [57]).

The variation of the material compressive strength is $\Delta\sigma_c = \sigma_c(0) - \sigma_c(c)$, where $\sigma_c(0) = \sigma_{cr}$ is the compressive strength of the material without cracks and $\sigma_c(c) \equiv \sigma_{cr}(c)$ is the compressive strength of the material with a crack. From Eq. (5) and Eq. of $\Delta\sigma_c$, with $P_{cr}(0) = 1$, results,

$$\Delta\sigma_c = \sigma_c(0) \cdot \left[1 - \left(1 - D_\sigma(c)\right)^{\frac{1}{\alpha+1}}\right].$$

Since the material (sandstone) behaves brittle (k = 1 and α = 1) this Eq. becomes, $\Delta\sigma_c = \sigma_c(0) \cdot \left[1 - \left(1 - D_\sigma(c)\right)^{0.5}\right]$, out of which we obtain the following equation for damage,

$$D_\sigma(a) = 1 - \left[1 - \frac{\Delta\sigma_c}{\sigma_c(0)}\right]^2. \qquad (24)$$

Table 3 presents the variation of compressive strength values obtained experimentally, $\Delta\sigma_{c,exp}$, and damage $D(c)$ calculated with Eq. (24) for different values of crack length, 2*c*.

Table 3. Variation of experimentally determined compressive strength on the basis of the data reported in [57] and deterioration values calculated with equation (24).

2*c* (mm)	5	10	15	20	25
$\Delta\sigma_{c,exp}$ (MPa)	30.76	65.38	88.46	115.38	126.92
$D_\sigma(c)$	0.283	0.546	0.688	0.821	0.866

Analyzing the data in Table 3 one can find how the material damage increases with the crack growth.

6. Conclusions

The criteria for strength and the relations for critical stresses in mechanical structures with cracks put forth in the present paper are based on the concept of specific energy and were the outcome of the principle of critical energy. This principle has made it possible to introduce deterioration in strength calculations.

There were established (on the basis of critical energy principle) failure criteria for sample and for structures with nonlinear behavior with cracks, *monotonic loaded*: - with normal stress; - with shear stress; - simultaneously, with a normal stress and a shear stress; - with several normal stresses and several shear stresses, as well as *cyclically loaded*: - with an ordinary normal stress block; - with a shear stress block; - simultaneously with a block of normal stress and one of shear stress.

The fatigue strength variation with crack length was analized in the case of fatigue loading (Fig. 2), as well as in the general case (Fig. 3).

The general theoretical results obtained in this study have been compared with particular results reported in the literature which fail to take into consideration the behavior of the sample material. The shear stresses calculated for samples with the first relation (18), compared with experimental results reported in the literature have shown good agreement between them. The calculation of deterioration in a real case is shown.

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