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Advanced notions and differential principles of motion in analytical dynamics

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Abstract. Using the author's researches, this paper extends the study by new formulations regarding the acceleration energies of first, second, third and fourth order. The advanced notions will be implemented in the differential equations of higher order, corresponding to suddenly and transitory motions.

Keywords. Acceleration energies, analytical dynamics, multibody systems, robotics.

1. Introduction

This paper is structured in three main parts. The first one is focused on a few new formulations, in the analytical dynamics of multibody mechanical systems (MBS), when they are characterized by suddenly movements (when the linear acceleration greater then *g* - gravitational acceleration), and the transitory motions. It demonstrates theoretical as well as experimental the existing of the acceleration energy of higher order. On the basis of the author's researches the acceleration energies of first, second, third and fourth orders will be presented in the both explicit and matrix formulation. The second part of this paper is devoted to present the advanced principles and motion equations in higher order, which will lead to variations in time of generalized forces, which dominating these types of mechanical systems. Within of these differential equations will be included the acceleration energy of higher order. The last part is an application in which the theoretical aspects presented in this paper are used to obtain the acceleration energies of higher order and the time variation laws for the generalized driving forces for a serial robot of type Fanuc.

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2. Advanced Notions in Analytical Mechanics

The phrase, entitled "advanced notions" founded in the analytical dynamics, is focused in this paper on the energies whose central functions are referring to the accelerations of higher order. They are developing in any suddenly and transitory motion of the mechanical systems. Leading to Appell's function, highlighted in 1899, also named "the kinetic energy of the accelerations" [14], the author has been developed a few new mathematical formulations on the expressions for acceleration energies of first, second, third and fourth order [4], [5],[7] – [12]. They will be presented, in explicit and matrix form, and the kinematical parameters will be expressed, using matrix exponentials [3], [4], [13], as well as matrix and differential transformations from advanced mechanics of the systems. The starting equation is:

$$E_A^{(p)i}=\frac{1}{2}\int \overset{(p)}{\bar{v}_i^T}\cdot\overset{(p)}{\bar{v}_i}\cdot dm=\frac{1}{2}\int Trace\left(\overset{(p+1)}{\bar{r}_i}\cdot\overset{(p+1)}{\bar{r}_i^{\,T}}\right)\cdot dm=\frac{1}{2}\cdot Trace\left(\overset{(p+1)}{\bar{r}_{C_i}}\cdot\overset{(p+1)}{\bar{r}_{C_i}^T}\right)\int dm+$$
$$+\frac{1}{2}\cdot Trace\left[\overset{(p+1)}{\bar{r}_{C_i}}\cdot\int {}^i\bar{r}_i^{*T}\cdot dm\cdot\overset{(p+1)}{{}_i^0[R]^T}\right]+\frac{1}{2}\cdot Trace\left[\overset{(p+1)}{{}_i^0[R]}\cdot\int {}^i\bar{r}_i^{*T}\cdot dm\cdot\overset{(p+1)}{\bar{r}_{C_i}^T}\right]+ \tag{1}$$
$$+\frac{1}{2}\cdot Trace\left[\int\overset{(p+1)}{{}_i^0[R]}\cdot{}^i\bar{r}_i^{*}\cdot{}^i\bar{r}_i^{*T}\cdot\overset{(p+1)}{{}_i^0[R]^T}\cdot dm\right]$$

$$E_A^{(p)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right]=\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[\int{}^i\bar{r}_i^{*}\cdot{}^i\bar{r}_i^{*T}\cdot dm+{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^T\cdot\int dm\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}+$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\cdot\int dm= \tag{2}$$
$$=\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[{}^iI_{pi}^{*}+M_i\cdot{}^i\bar{r}_{C_i}\cdot{}^i\bar{r}_{C_i}^T\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\bar{p}_i}\cdot\overset{(p+1)}{\bar{p}_i^T}\right]\cdot M_i$$

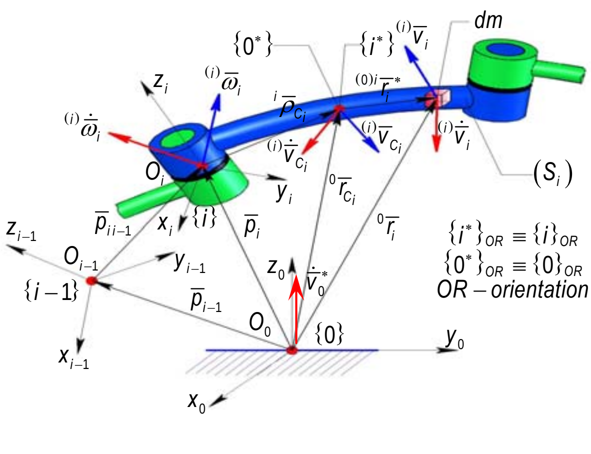


Fig. 1 A kinetic ensemble from MBS

Considering the symbols from Fig.1, refer to (1) it defines the acceleration energy of order $"p = 1,2,3,..."$ for a kinetic ensemble $i = 1 \rightarrow n$, belonging to MBS, while refer to (2) it is corresponding to whole mechanical system. The symbols, included in (1), (2) and Fig.1 have the significances: (p) and $(p+1)$ is the order of the absolute and time derivatives; ${}^{i}\overline{r}_i^{*}$ the position vector of the elementary mass dm relative to reference frame $\{i^*\}$ applied in the mass center; ${}^{(i)}\overline{r}_{C_i}$ the position vector of the mass center projected on $\{0\}$ or $\{i\}$ reference frame; ${}^{0}_{i}[R]$ the rotation matrix between the two frames above mentioned. Whereas MBS is characterized by n degrees of freedom (d. o. f.) also named generalized coordinates, they are included in the column matrix $\overline{\theta}(t) = (q_i(t),\ for\ \text{i=1} \rightarrow \text{n})^T$, as well their time derivatives. Using (1) and (2), in the following of the first chapter, the expressions of the acceleration energies of first, second, third and fourth orders will be demonstrated and presented, in explicit and matrix form.

2.1 Acceleration Energy of First Order

According to the scientific literature [1], [2], [4]-[12], [14] and [15] in 1879, Gibbs defines the differential equations of motion, on which, in 1899, Paul Appell performs a detailed study. As a result of this study, were deduced the equations, known as Gibbs-Appell, which are applied for holonomic and nonholonomic systems, where the role of the kinetic energy was substituted by the acceleration energy also known as Appell's function or "kinetic energy of acceleration" [14]. Unlike the studies, above mentioned, in the paper [4] and not only, the author, of this paper, was established the acceleration energy in a generalized form, corresponding to a MBS and it was named acceleration energy of first order. Starting from (1) and (2), equation for defining the acceleration energy of first order is:

$$\begin{aligned} E_A^{(1)i} &= \frac{1}{2} \cdot \int \dot{\overline{v}}_i^T \cdot \dot{\overline{v}}_i \cdot dm = \frac{1}{2} \cdot \int Trace\left(\ddot{\overline{r}}_i \cdot \ddot{\overline{r}}_i^T \right) \cdot dm = \\ &= \frac{1}{2} \cdot \int Trace\left[\left(\ddot{\overline{r}}_{C_i} + {}^{0}_{i}[\ddot{R}] \cdot {}^{i}\overline{r}_i^{*} \right) \cdot \left(\ddot{\overline{r}}_{C_i}^T + {}^{i}\overline{r}_i^{*T} \cdot {}^{0}_{i}[\ddot{R}]^T \right) \right] \cdot dm \end{aligned} \tag{3}$$

After significant matrix and differential transformations in (1) and (2), the author was obtained the expression of definition, in explicit form, for acceleration energy of first order and corresponding to whole mechanical system (MBS):

$$\begin{aligned} E_A^{(1)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t)\right] &= (-1)^{\Delta_m} \cdot \frac{1-\Delta_m}{1+3\cdot\Delta_m} \sum_{i=1}^{n} \left[\frac{1}{2} \cdot M_i \cdot {}^{(i)}\dot{\overline{v}}_{C_i}^T \cdot {}^{(i)}\dot{\overline{v}}_{C_i} \right] + \\ &+ \Delta_m^2 \cdot \sum_{i=1}^{n} \frac{1}{2} \cdot {}^{(i)}\dot{\overline{\omega}}_i^T \cdot {}^{(i)}I_i^{*} \cdot {}^{(i)}\dot{\overline{\omega}}_i + \Delta_m^2 \cdot \sum_{i=1}^{n} \left[{}^{(i)}\dot{\overline{\omega}}_i^T \cdot \left({}^{(i)}\overline{\omega}_i \times {}^{(i)}I_i^{*} \cdot {}^{(i)}\overline{\omega}_i \right) \right] + \\ &+ \Delta_m^2 \cdot \sum_{i=1}^{n} \left\{ \frac{1}{2} \cdot {}^{(i)}\overline{\omega}_i^T \cdot \left[{}^{(i)}\overline{\omega}_i^T \cdot Trace\left({}^{(i)}I_{pi}^{*} \right) \cdot {}^{(i)}\overline{\omega}_i - {}^{(i)}\overline{\omega}_i^T \cdot {}^{(i)}I_{pi}^{*} \cdot {}^{(i)}\overline{\omega}_i \right] \cdot {}^{(i)}\overline{\omega}_i \right\} \end{aligned} \tag{4}$$

Refer to equation (4), it includes the following symbols and parameters: $\Delta_M = \{\{-1; general\ motion\}; \{0; translation\}; \{1; rotation\}\}$; ${}^{(i)}\dot{\bar{v}}_{C_i}$ the absolute acceleration of the mass center; ${}^{(i)}\bar{\omega}_i$ and ${}^{(i)}\dot{\bar{\omega}}_i$ the absolute angular velocity and acceleration of the kinetic ensemble (i) from Fig.1; M_i, ${}^{(i)}I_i^*$ and ${}^{(i)}I_{pi}^*$ represent the mass, the axial and centrifugal inertial tensor, as well as the planar centrifugal inertial tensor corresponding to the entire kinetic ensemble (i), relative to the mass center C_i:

$$ {}^{(i)}I_i^* = \int \left({}^{(i)}\bar{r}_i^* \times\right)\left({}^{(i)}\bar{r}_i^* \times\right)^T dm, \qquad {}^{(i)}I_{pi}^* = \int {}^{(i)}\bar{r}_i^* \cdot {}^{(i)}\bar{r}_i^{*T} dm. \tag{5} $$

The same acceleration energy of first order, above written by (4) and corresponding to MBS, the author has developed [4], [6] a matrix expression, and below presented:

$$ \begin{aligned} E_A^{(1)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right] = \frac{1}{2}\cdot\{\ddot{\bar{\theta}}^T(t)\cdot M[\bar{\theta}(t)]\cdot\ddot{\bar{\theta}}(t) + \\ +\ddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] + \left[\dot{\bar{\theta}}^T(t)\cdot D\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\cdot\dot{\bar{\theta}}(t)\right]\} \end{aligned} \tag{6} $$

Refer to the equation (6), it contains on the one hand the first and second time derivatives of the column matrix $\bar{\theta}(t)$ of the generalized variables, also named generalized velocities and accelerations. On the other hand in (6) a set of dynamics matrices are founded, whose expressions are defined below:

$$ M\left[\bar{\theta}(t)\right] = \underset{(n\times n)}{Matrix}\left\{ M_{ij} = \sum_{k=\max(i;j)}^{n} Trace\left[A_{ki}\cdot {}^kI_{psk}\cdot A_{kj}^T\right] \quad \begin{matrix} i=1\to n \\ j=1\to n \end{matrix} \right\}, \tag{7} $$

$$ V\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] = \underset{(n\times 1)}{Matrix}\left[V_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right];\ i=1\to n\right], \tag{8} $$

$$ V_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] = \dot{\bar{\theta}}^T\cdot\left[V_{ijm} = \sum_{k=\max(j;m)}^{n} Trace\left[A_{ki}\cdot {}^kI_{psk}\cdot A_{kjm}^T\right] \quad \begin{matrix} j=1\to n \\ m=1\to n \end{matrix} \right]\cdot\dot{\bar{\theta}}, \tag{9} $$

$$ D\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right] = \underset{(n\times n)}{Matrix}\left\{ \begin{matrix} D_{ij} = \dot{\bar{\theta}}^T\cdot\left[\begin{matrix} D_{ijlm} = \sum_{k=\max(i;j;l;m)}^{n} Trace\left[A_{kij}\cdot {}^kI_{psk}\cdot A_{klm}^T\right] \\ where\ rows: l=1\to n\ \ and\ columns: m=1\to n \end{matrix} \right]\cdot\dot{\bar{\theta}} \\ where\quad rows: i=1\to n\quad and\quad columns: j=1\to n \end{matrix} \right\} \tag{10} $$

Equation (7) is the mass matrix or inertia matrix of acceleration energies, refer to (8) with (9) it represents the column matrix of centrifugal and Coriolis terms, while refer to (10), it is named pseudo-inertial matrix of the acceleration energy. All three matrices are known, for example [1], [4]-[12]. Their components illustrate on the one hand the mass properties included in the pseudo-inertial tensor:

$$ \underset{(4\times 4)}{{}^kI_{psk}} = \begin{bmatrix} \int {}^k\bar{r}_k\cdot{}^k\bar{r}_k^T\cdot dm & \int {}^k\bar{r}_k\cdot dm \\ \int {}^k\bar{r}_k^T\cdot dm & \int dm \end{bmatrix} = \left[\begin{array}{c|c} {}^kI_{pk} & M_k\cdot{}^k\bar{r}_{C_k} \\ \hline M_k\cdot{}^k\bar{r}_{C_k}^T & M_k \end{array}\right]. \tag{11} $$

On the other hand, the dynamics matrices include the so called differential matrices of first and second order, which correspond to the homogeneous transformations matrices between the reference systems of the MBS, see Fig. 1. So, the differential matrix of first order ($\overline{p}$ -position, R -orientation) shown as:

$$\underset{(4\times4)}{A_{ki(j)}} = \left[\begin{array}{ccc:c} & A_{ki(j)}(R) & & A_{ki(j)}(\overline{p}) \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \tag{12}$$

$$A_{ij}(R) = \frac{\partial}{\partial q_j}\{{}^0_i[R]\} = \left\{\exp\left\{\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right\}\right\}\cdot\left(\overline{k}_j^{(0)}\times\right)\cdot\Delta_j\cdot\exp\left\{\sum_{l=j}^{i}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\cdot R_{i0}^{(0)}, \tag{13}$$

$$\underset{(3\times1)}{A_{ij}(\overline{p})} = \frac{\partial\,\overline{p}_i}{\partial q_j} = \left\{\exp\left[\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\right\}\cdot X_j + \exp\left[\sum_{l=j}^{i}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right]\cdot\overline{p}_i^{(0)} + A_{ij}^{*}(\overline{p}) \tag{14}$$

$$\textit{while}\;\; X_j = \left(\overline{p}_j^{(0)}\times\right)\cdot\overline{k}_j^{(0)}\cdot\Delta_j + \left(1-\Delta_j\right)\cdot\overline{k}_j^{(0)}; \tag{15}$$

$$A_{ij}^{*}(\overline{p}) = \Delta_j\cdot\exp\left[\sum_{k=0}^{j-1}\left(\overline{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\cdot\sum_{l=j}^{i}\left\{\exp\left[\sum_{m=j-1}^{l-1}\left(\overline{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\cdot\delta_m\right]\right\}\cdot\overline{b}_l \tag{16}$$

The differential matrix of second order is defined with matrix exponential functions:

$$A_{ijk}(R) = \left[\begin{array}{ccc:c} & A_{ijk}(R) & & A_{ijk}(\overline{p}) \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \tag{17}$$

$$A_{ijk}(R) = \frac{\partial^2}{\partial q_j\cdot\partial q_k}\{{}^0_i[R]\} = \left\{\exp\left\{\sum_{l=0}^{k-1}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot\left(\overline{k}_k^{(0)}\times\right)\cdot\Delta_k\cdot A_{ijk}^{*}(R) \tag{18}$$

$$A_{ijk}^{*}(R) = \left\{\exp\left\{\sum_{m=k}^{j-1}\left(\overline{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\right\}\right\}\cdot\left(\overline{k}_m^{(0)}\times\right)\cdot\Delta_m\cdot\left\{\exp\left\{\sum_{p=m}^{i}\left(\overline{k}_p^{(0)}\times\right)\cdot q_p\cdot\Delta_p\right\}\right\}\cdot R_{i0}^{(0)} \tag{19}$$

$$A_{ijk}(\overline{p}) = \frac{\partial^2\,\overline{p}_i}{\partial q_j\cdot\partial q_k} = \frac{\partial}{\partial q_k}\left[A_{ij}(\overline{p})\right], \text{ and } A_{ij}(\overline{p}) \text{ is given by (14) – (16).}$$

According to [3]-[11], the sub-matrices from (12) and (17) are determined using matrix exponential functions. In their expressions the following symbols are included as follows: $\overline{k}_{k(j)}^{(0)}$ is named the unit vector, in the initial configuration, of the axis corresponding to the generalized coordinate $q_{k(j)}$, while $\Delta_{k(j)} = 1$ when $q_{k(j)}$ angular coordinate and this is $\Delta_{k(j)} = 0$ otherwise. The terms $\overline{p}_i^{(0)}$ and $R_{i0}^{(0)}$ represent the position vector, respectively the orientation matrix of the system $\{i\}$ in relation to $\{0\}$, in the initial configuration.

In conclusion of this section, refer to (4), it can be seen that a generalization of König's theorem from analytical dynamics but this is extended on the acceleration energies of first order, for anything mechanical system.

2.2 Acceleration Energy of Second Order

According to the research of the author [5], [8] – [12], the suddenly motion of MBS, the transient motion phases, as well as the mechanical systems subjected to the action of a system of external forces, with a time variation law, are characterized by linear and angular accelerations of higher order. A simple example in agreement with this statement is the simplified mechanical system shown in Fig. 2. Therefore, in this section the acceleration energy of second order is developed in explicit and matrix form. Starting from (1) and (2), the acceleration energy of second order becomes:

$$\begin{aligned} E_A^{(2)i} &= \frac{1}{2}\int \ddot{\overline{v}}_i^{\,T} \cdot \ddot{\overline{v}}_i \cdot dm = \frac{1}{2}\int Trace\left(\dddot{\overline{r}}_i \cdot \dddot{\overline{r}}_i^{\,T}\right)\cdot dm = \\ &= \frac{1}{2}\cdot Trace\left(\dddot{\overline{r}}_{C_i} \cdot \dddot{\overline{r}}_{C_i}^{\,T}\right)\int dm + \frac{1}{2}\cdot Trace\left[\dddot{\overline{r}}_{C_i} \cdot \int {}^i\overline{r}_i^{\,*T} \cdot dm \cdot {}_i^0[\dddot{R}]^T\right] + \\ &+ \frac{1}{2}\cdot Trace\left[{}_i^0[\dddot{R}]\cdot \int {}^i\overline{r}_i^{\,*T} \cdot dm \cdot \dddot{\overline{r}}_{C_i}^{\,T}\right] + \frac{1}{2}\cdot Trace\left[\int {}_i^0[\dddot{R}]\cdot {}^i\overline{r}_i^{\,*} \cdot {}^i\overline{r}_i^{\,*T} \cdot {}_i^0[\dddot{R}]^T \cdot dm\right] \end{aligned} \tag{20}$$

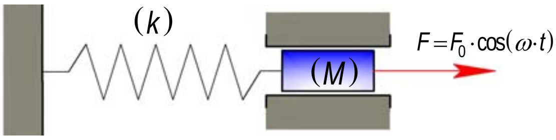


Fig. 2 Simplified mechanical system

According to [8]-[10], applying a few matrix and differential transformations, the explicit form of the acceleration energy of second order for a MBS is obtained as:

$$\begin{aligned} & E_A^{(2)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right] = (-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot {}^i\ddot{\overline{v}}_{C_i}^{\,T}\cdot {}^i\ddot{\overline{v}}_{C_i}\right\} + \\ & +\Delta_m^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot {}^i\ddot{\overline{\omega}}_i^{\,T}\cdot {}^iI_i^*\cdot {}^i\ddot{\overline{\omega}}_i + 2\cdot {}^i\overline{\omega}_i^{\,T}\cdot\left({}^i\ddot{\overline{\omega}}_i\times {}^iI_{pi}^*\cdot {}^i\dot{\overline{\omega}}_i\right) + \right. \\ & \left. + {}^i\dot{\overline{\omega}}_i^{\,T}\cdot\left({}^i\ddot{\overline{\omega}}_i\times {}^iI_{pi}^*\cdot {}^i\overline{\omega}_i\right) - {}^i\overline{\omega}_i^{\,T}\cdot\left({}^i\ddot{\overline{\omega}}_i^{\,T}\cdot {}^iI_i^*\cdot {}^i\overline{\omega}_i\right)\cdot {}^i\overline{\omega}_i\right\} + \\ & +\Delta_m^2\cdot\sum_{i=1}^{n}\left\{2\cdot {}^i\dot{\overline{\omega}}_i^{\,T}\cdot\left({}^i\overline{\omega}_i^{\,T}\cdot {}^iI_i^*\cdot {}^i\dot{\overline{\omega}}_i\right)\cdot {}^i\overline{\omega}_i + 2\cdot {}^i\overline{\omega}_i^{\,T}\cdot\left[{}^i\dot{\overline{\omega}}_i^{\,T}\cdot {}^iI_{pi}^*\cdot {}^i\dot{\overline{\omega}}_i\right]\cdot {}^i\overline{\omega}_i - \right. \\ & -5\cdot\left({}^i\dot{\overline{\omega}}_i^{\,T}\cdot {}^iI_{pi}^*\right)\cdot\left({}^i\dot{\overline{\omega}}_i^{\,T}\cdot {}^i\overline{\omega}_i\right)\cdot {}^i\overline{\omega}_i + \\ & +\frac{5}{2}\cdot\left({}^i\dot{\overline{\omega}}_i^{\,T}\cdot {}^i\overline{\omega}_i\right)\cdot Trace\left({}^iI_{pi}^*\right)\cdot\left({}^i\dot{\overline{\omega}}_i^{\,T}\cdot {}^i\overline{\omega}_i\right) + \frac{1}{2}\cdot {}^i\dot{\overline{\omega}}_i^{\,T}\cdot\left[{}^i\overline{\omega}_i^{\,T}\cdot {}^iI_{pi}^*\cdot {}^i\overline{\omega}_i\right]\cdot {}^i\dot{\overline{\omega}}_i + \\ & \left. + {}^i\overline{\omega}_i^{\,T}\cdot\left[{}^i\overline{\omega}_i^{\,T}\cdot\left({}^i\dot{\overline{\omega}}_i\times {}^iI_{pi}^*\cdot {}^i\overline{\omega}_i\right)\right]\cdot {}^i\overline{\omega}_i + \frac{1}{2}\cdot {}^i\overline{\omega}_i^{\,T}\cdot\left[{}^i\overline{\omega}_i^{\,T}\cdot\left({}^i\overline{\omega}_i^{\,T}\cdot {}^iI_i^*\cdot {}^i\overline{\omega}_i\right)\cdot {}^i\overline{\omega}_i\right]\cdot {}^i\overline{\omega}_i\right\}. \end{aligned} \tag{21}$$

It can be also considered an extension of the generalization of König's theorem of the second order. Refer to (20) and (21) they include the symbols and parameters specified in the second and third pages. At these, the terms which are function of

$\dddot{\theta} = (\dddot{q}_i,\ for\ \ i=1 \to n)^T$, representing the generalized accelerations of second order are also added. According to [8] – [10], following of the application a few of matrix and differential transformations, the matrix expression of the acceleration energy of second order is determined as follows:

$$
\begin{aligned}
&E_A^{(2)}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t);\dddot{\bar{\theta}}(t)\right]= \\
&=\frac{1}{2}\cdot\dddot{\bar{\theta}}^T(t)\cdot M\left[\bar{\theta}(t)\right]\cdot\dddot{\bar{\theta}}(t)+3\cdot\dddot{\bar{\theta}}^T(t)\cdot V\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]+ \\
&+\dddot{\bar{\theta}}^T(t)\cdot H\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\cdot\dot{\bar{\theta}}(t)+\frac{9}{2}\cdot\ddot{\bar{\theta}}^T(t)\cdot D\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]\cdot\dot{\bar{\theta}}(t)+ \\
&+3\cdot\ddot{\bar{\theta}}^T(t)\cdot K\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]+\frac{1}{2}\cdot\dot{\bar{\theta}}^T(t)\cdot N\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]\cdot\dot{\bar{\theta}}(t)
\end{aligned}
\tag{22}
$$

Alongside (7), the other five dynamics matrices are included in the acceleration energy of second order (22). They are defined below in a symbolically form:

$$
\begin{aligned}
&V\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]= \\
&=\underset{(n\times1)}{Matrix}\left\{V_i^*\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]=\ddot{\bar{\theta}}^T\cdot\left[\left\{V_{ijm}=V_{imj}\right\}\ \begin{matrix} j=1\to n \\ m=1\to n\end{matrix}\right]\cdot\dot{\bar{\theta}},\quad i=1\to n\right\}
\end{aligned}
\tag{23}
$$

$$
H\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]=\underset{(n\times n)}{Matrix}\left\{H_{ij}\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]\ \begin{matrix} i=1\to n \\ j=1\to n\end{matrix}\right\},
\tag{24}
$$

$$
H_{ij}\left[\bar{\theta}(t);\dot{\bar{\theta}}^2(t)\right]=\dot{\bar{\theta}}^T\cdot\left\{H_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlm}^T\right]\ \begin{matrix} l=1\to n \\ m=1\to n\end{matrix}\right\}\cdot\dot{\bar{\theta}};
$$

$$
D\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\ddot{\bar{\theta}}(t)\right]=\underset{(n\times n)}{Matrix}\left\{D_{ij}\ \begin{matrix} i=1\to n \\ j=1\to n\end{matrix}\right\},
\tag{25}
$$

$$
D_{ij}=\ddot{\bar{\theta}}^T\cdot\left[D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Trace\left[A_{kij}\cdot{}^kI_{psk}\cdot A_{klm}^T\right]\ \begin{matrix} l=1\to n \\ m=1\to n\end{matrix}\right]\cdot\dot{\bar{\theta}};
$$

$$
K\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]=\underset{(n\times1)}{Matrix}\left\{\bar{K}_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]\quad i=1\to n\right\},
\tag{26}
$$

$$
\begin{aligned}
&\bar{K}_i\left[\bar{\theta}(t);\dot{\bar{\theta}}^4(t)\right]= \\
&=\dot{\bar{\theta}}^T\cdot\left\{\begin{matrix}\dot{\bar{\theta}}^T\cdot\left[K_{ijlmp}=\displaystyle\sum_{k=\max(i;j;l;m;p)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlmp}^T\right]\ \begin{matrix} m=1\to n \\ p=1\to n\end{matrix}\right]\cdot\dot{\bar{\theta}} \\ j=1\to n;\ l=1\to n\end{matrix}\right\}\cdot\dot{\bar{\theta}};
\end{aligned}
$$

$$N\left[\overline{\theta}(t);\dot{\overline{\theta}}^4(t)\right] = \underset{(n\times n)}{Matrix}\left\{N_{ij} \quad \begin{matrix} i=1\to n \\ j=1\to n \end{matrix}\right\}, \tag{27}$$

$$N_{ij} = \dot{\overline{\theta}}^T \cdot \left\{ \begin{matrix} \dot{\overline{\theta}}^T \cdot \left[N_{ijlmpr} = \sum_{k=\max(i;j;l;m;p;r)}^{n} Tr\left[A_{kijl} \cdot {}^k I_{psk} \cdot A_{kmpr}^T\right] \quad \begin{matrix} p=1\to n \\ r=1\to n \end{matrix} \right] \cdot \dot{\overline{\theta}} \\ l=1\to n \quad m=1\to n \end{matrix} \right\} \cdot \dot{\overline{\theta}} .$$

Equations (23) - (27) are dynamics matrices of second order with mass and inertia properties. The differential matrix of third order, component of the dynamics matrix (24), with (26) and (27) has the form:

$$\underset{(4\times 4)}{A_{ijkm}} = \left[\begin{array}{ccc|c} & A_{ijkm}(R) & & A_{ijkm}(\overline{p}) \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{28}$$

According to [3]-[13], the sub-matrices from (28) are expressed by means of the matrix exponentials functions:

$$A_{ijkm}(R) = \left\{ exp\left\{ \sum_{l=0}^{m-1} \left(\overline{k}_l^{(0)} \times\right) \cdot q_l \cdot \Delta_l \right\} \right\} \cdot \left(\overline{k}_m^{(0)} \times\right) \cdot \Delta_m \cdot A_{ijkm}^*(R) \tag{29}$$

$$A_{ijkm}^*(R) = \left\{ exp\left\{ \sum_{p=m}^{k-1} \left(\overline{k}_p^{(0)} \times\right) \cdot q_p \cdot \Delta_p \right\} \right\} \cdot \left(\overline{k}_p^{(0)} \times\right) \cdot \Delta_p \cdot A_{ijkm}^{**}(R), \tag{30}$$

$$A_{ijkm}^{**}(R) = \left\{ exp\left\{ \sum_{r=p}^{j-1} \left(\overline{k}_r^{(0)} \times\right) \cdot q_r \cdot \Delta_r \right\} \right\} \cdot \left(\overline{k}_r^{(0)} \times\right) \cdot \Delta_r \cdot exp\left\{ \sum_{s=r}^{i} \left(\overline{k}_s^{(0)} \times\right) \cdot q_s \cdot \Delta_s \right\} \tag{31}$$

$$\text{and } A_{ijkm}(\overline{p}) = \frac{\partial}{\partial q_k}\left[A_{ijk}(\overline{p})\right] = \frac{\partial^2}{\partial q_k \cdot \partial q_m}\left[A_{ij}(\overline{p})\right], \tag{32}$$

where the column matrix $A_{ij}(\overline{p})$ is given by equations (14) – (16).

The differential matrix of fourth order, component of the dynamics matrices (26) has the following definition expression:

$$\underset{(4\times 4)}{A_{ijkmp}} = \left[\begin{array}{ccc|c} & A_{ijkmp}(R) & & A_{ijkmp}(\overline{p}) \\ \hline 0 & 0 & 0 & 0 \end{array}\right]; \tag{33}$$

$$A_{ijkmp}(R) = \frac{\partial^4}{\partial q_j \cdot \partial q_k \cdot \partial q_m \cdot \partial q_p}\left\{ {}^0_i[R] \right\} = \left\{ \exp\left\{ \sum_{l=0}^{p-1} \left(\overline{k}_l^{(0)} \times\right) \cdot q_l \cdot \Delta_l \right\} \right\} \cdot A_{ijkmp}^*(R), \tag{34}$$

$$A_{ijkmp}^*(R) = \left(\overline{k}_p^{(0)} \times\right) \cdot \Delta_p \cdot \left\{ \exp\left\{ \sum_{r=p}^{m-1} \left(\overline{k}_p^{(0)} \times\right) \cdot q_r \cdot \Delta_r \right\} \right\} \cdot A_{ijkmp}^{**}(R),$$

$$A_{ijkmp}^{**}(R) = \left(\overline{k}_r^{(0)} \times\right) \cdot \Delta_r \cdot \left\{ \exp\left\{ \sum_{s=r}^{k-1} \left(\overline{k}_r^{(0)} \times\right) \cdot q_s \cdot \Delta_s \right\} \right\} \cdot A_{ijkmp}^{***}(R),$$

$$A^{***}_{ijkmp}(R)=\left(\overline{k}^{(0)}_{s}\times\right)\cdot\Delta_s\cdot\left\{\exp\left\{\sum_{u=s}^{j-1}\left(\overline{k}^{(0)}_{u}\times\right)\cdot q_u\cdot\Delta_u\right\}\right\}\cdot A^{****}_{ijkmp}(R),$$

$$A^{****}_{ijkmp}(R)=\left(\overline{k}^{(0)}_{u}\times\right)\cdot\Delta_u\cdot\exp\left\{\sum_{v=u}^{i}\left(\overline{k}^{(0)}_{v}\times\right)\cdot q_v\cdot\Delta_v\right\}\cdot R^{(0)}_{i0};$$

$$A_{ijkmp}(\overline{p})=\frac{\partial A_{ijkm}(\overline{p})}{\partial q_p}=\frac{\partial^2 A_{ijkm}(\overline{p})}{\partial q_m\cdot\partial q_p}=\frac{\partial^3 A_{ij}(\overline{p})}{\partial q_k\cdot\partial q_m\cdot\partial q_p}=\frac{\partial^4\overline{p}_i}{\partial q_j\cdot\partial q_k\cdot\partial q_m\cdot\partial q_p}. \tag{35}$$

The components of the differential matrix of fourth order, included in (34), according to same [3]-[13], it observes that they have been also determined by the matrix exponentials functions.

2.3 Acceleration Energy of Third Order

Considering the introductory aspects from previous section, the suddenly motion of MBS, the transient motion phases, as well as the mechanical systems subjected to the action of a system of external forces with a time variation law, in which the robots are included, the dynamic study is extended on the acceleration energy of third order. As example, the same Fig.2 can be also considered. So, using (1) and (2), in this case, in accordance with [9] – [12], the author proposes the equation of the acceleration energy of third order. This will be defined in both variants: explicit and matrix form. First of all, the starting equation shows as:

$$\begin{aligned}E^{(3)}_{A}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t);\ddddot{\overline{\theta}}(t)\right]&=\frac{1}{2}\sum_{i=1}^{n}\int\dddot{\overline{v}}_i^T\cdot\dddot{\overline{v}}_i\cdot dm=\frac{1}{2}\sum_{i=1}^{n}\int Trace\left(\ddddot{\overline{r}}_i\cdot\ddddot{\overline{r}}_i^T\right)\cdot dm=\\&=\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}^0_i\left[\dddot{R}\right]\cdot\left[{}^iI^*_{pi}+M_i\cdot{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}^T_{C_i}\right]\cdot{}^0_i\left[\dddot{R}\right]^T\right\}+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\ddddot{\overline{p}}_i\cdot\ddddot{\overline{p}}^T_i\right]\cdot M_i\end{aligned} \tag{36}$$

where $\ddddot{\overline{r}}_i$ represents the absolute acceleration of third order of the elementary mass dm, and $\ddddot{\overline{p}}_i$ expresses the absolute acceleration of third order of the origin $O_i\in\{i\}$, where $\{i\}$ is the reference frame, in accordance with the same Fig.1. According to [12], applying a few matrix and differential transformations the acceleration energy of third order for MBS is obtained, under the explicit form, in the two variants:

$$\begin{aligned}E^{(3)}_{A}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t);\ddddot{\overline{\theta}}(t)\right]&=(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot{}^i\dddot{\overline{v}}^T_{C_i}\cdot{}^i\dddot{\overline{v}}_{C_i}\right\}+\\&+\Delta^2_m\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^i\dddot{\overline{\omega}}^T_i\cdot{}^iI^*_i\cdot{}^i\dddot{\overline{\omega}}_i+3\cdot\overline{\omega}^T_i\cdot\left(\dddot{\overline{\omega}}_i\times I^*_{pi}\cdot\ddot{\overline{\omega}}_i\right)+3\cdot\dot{\overline{\omega}}^T_i\cdot\left(\dddot{\overline{\omega}}_i\times I^*_{pi}\cdot\dot{\overline{\omega}}_i\right)+\right.\\&+3\cdot\ddot{\overline{\omega}}^T_i\cdot\left(\dddot{\overline{\omega}}_i\times I^*_{pi}\cdot\overline{\omega}_i\right)+2\cdot\left(\overline{\omega}_i\times\dddot{\overline{\omega}}_i\right)^T\cdot I^*_{pi}\cdot\left(\dot{\overline{\omega}}_i\times\overline{\omega}_i\right)-5\cdot\dot{\overline{\omega}}^T_i\cdot\left[\overline{\omega}^T_i\cdot I^*_i\cdot\dddot{\overline{\omega}}_i\right]\cdot\overline{\omega}_i-\\&\left.-\overline{\omega}^T_i\cdot\left[\dot{\overline{\omega}}^T_i\cdot I^*_i\cdot\dddot{\overline{\omega}}_i\right]\cdot\overline{\omega}_i+\overline{\omega}^T_i\cdot\left[\overline{\omega}^T_i\cdot I^*_{pi}\cdot\left(\dddot{\overline{\omega}}_i\times\overline{\omega}_i\right)\right]\cdot\overline{\omega}_i\right\}\end{aligned} \tag{37}$$

First variant, of the acceleration energy of third order, under the explicit form, is presented by (37), and the second variant, highlighting the generalized variables, is also shown in the explicit form thus:

$$E_A^{(3)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t);\ddddot{\overline{\theta}}(t)\right]=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}M_{ij}\cdot\ddddot{q}_i\cdot\ddddot{q}_j+$$
$$+4\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}V_{ijm}\cdot\ddddot{q}_j\cdot\ddot{q}_k\cdot\dot{q}_m+3\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}V_{ijm}^{*}\cdot\ddddot{q}_j\cdot\ddot{q}_k\cdot\ddot{q}_m+ \tag{38}$$
$$+6\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}H_{ijlm}\ddddot{q}_j\cdot\ddot{q}_k\cdot\dot{q}_l\cdot\dot{q}_m+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{p=1}^{n}K_{ijlmp}\cdot\ddddot{q}_j\cdot\dot{q}_k\cdot\dot{q}_l\cdot\dot{q}_m\cdot\dot{q}_p\ ;$$

$$E_A^{(3)}(t)=E_A^{(3)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t);\ddddot{\overline{\theta}}(t)\right]+E_A^{(3)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right] \tag{39}$$

Equation (39) contains the two terms corresponding to complete form. Refer to (37) and (38), they include the symbols and parameters specified in the first two sections of this chapter. At these, only the terms which are a function of $\ddddot{\overline{\theta}}=(\ddddot{q}_i,\ for\ \text{i=1}\rightarrow\text{n})^T$, representing the generalized accelerations of third order are added. Similarly with the first two types of acceleration energies, it can also observe an extension of the generalization of König's theorem regarding the acceleration energy of third order.

According to [10] and [12], following of the application a few of matrix and differential transformations, the matrix expression of the acceleration energy of third order is defined as function of $\ddddot{\overline{\theta}}=(\ddddot{q}_i,\ for\ \text{i=1}\rightarrow\text{n})^T$ with:

$$E_A^{(3)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t);\ddddot{\overline{\theta}}(t)\right]=\frac{1}{2}\cdot\ddddot{\overline{\theta}}^T(t)\cdot M\left[\overline{\theta}(t)\right]\cdot\ddddot{\overline{\theta}}(t)+$$
$$+4\cdot\ddddot{\overline{\theta}}^T(t)\cdot V\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right]+3\cdot\ddddot{\overline{\theta}}^T(t)\cdot V^{*}\left[\overline{\theta}(t);\ddot{\overline{\theta}}^2(t)\right]+ \tag{40}$$
$$+6\cdot\ddddot{\overline{\theta}}^T(t)\cdot H^{*}\left[\overline{\theta}(t);\dot{\overline{\theta}}^2(t)\right]\cdot\ddot{\overline{\theta}}(t)+\ddddot{\overline{\theta}}^T(t)\cdot K^{*}\left[\overline{\theta}(t);\dot{\overline{\theta}}^4(t)\right]$$

The dynamics matrices of third order, included in the equation (40), have the following expressions of definition:

$$V\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\dddot{\overline{\theta}}(t)\right]=\underset{(n\times1)}{Matrix}\left\{\dddot{\overline{\theta}}^T\cdot\begin{bmatrix}V_{ijm} & \begin{matrix}j=1\rightarrow n\\ m=1\rightarrow n\end{matrix}\end{bmatrix}\cdot\dot{\overline{\theta}}\quad where\quad i=1\rightarrow n\right\}^T \tag{41}$$

$$V^{*}\left[\overline{\theta}(t);\ddot{\overline{\theta}}^2(t)\right]=\underset{(n\times1)}{Matrix}\left\{\ddot{\overline{\theta}}^T\cdot\begin{bmatrix}V_{ijm} & \begin{matrix}j=1\rightarrow n\\ m=1\rightarrow n\end{matrix}\end{bmatrix}\cdot\ddot{\overline{\theta}}\quad where\quad i=1\rightarrow n\right\}^T \tag{42}$$

$$H^{*}\left[\overline{\theta}(t);\dot{\overline{\theta}}^2(t)\right]=\underset{(n\times n)}{Matrix}\left\{H_{ij}^{*}\left[\overline{\theta}(t);\dot{\overline{\theta}}^2(t)\right]\quad \begin{matrix}i=1\rightarrow n\\ j=1\rightarrow n\end{matrix}\right\}, \tag{43}$$

$$H_{ij}^{*}\left[\overline{\theta}(t);\dot{\overline{\theta}}^2(t)\right]=\dot{\overline{\theta}}^T\cdot\begin{bmatrix}H_{ijlm} & \begin{matrix}l=1\rightarrow n\\ m=1\rightarrow n\end{matrix}\end{bmatrix}\cdot\dot{\overline{\theta}}\ ;$$

$$K^*\left[\overline{\theta}(t);\dot{\overline{\theta}}^4(t)\right]=\underset{(n\times 1)}{Matrix}\left\{K_i^*\left[\overline{\theta}(t);\dot{\overline{\theta}}^4(t)\right]\quad where\ i=1\rightarrow n\right\}^T, \tag{44}$$

$$K_i^*\left[\overline{\theta}(t);\dot{\overline{\theta}}^4(t)\right]=\dot{\overline{\theta}}^T\cdot\left\{\begin{matrix}\dot{\overline{\theta}}^T\cdot\left[K_{ijlmp}\ \begin{matrix}m=1\rightarrow n\\ p=1\rightarrow n\end{matrix}\right]\cdot\dot{\overline{\theta}}\\ j=1\rightarrow n\ ;\ l=1\rightarrow n\end{matrix}\right\}\cdot\dot{\overline{\theta}}\ .$$

The last two have the same matrix form with (24) and (26). All components of the above dynamics matrices are also determined with matrix exponentials described in the previous section, according to [10] and [12].

2.4 Acceleration Energy of Fourth Order

On the basis of the expressions (1) and (2), the researches of the author are extended about the acceleration energy of fourth order. The starting equation is shown below:

$$\left\{\begin{gathered}E_A^{(4)}\left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\cdots;\overset{(5)}{\overline{\theta}}(t)\right]=\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(5)}{\overline{p}_i}\cdot\overset{(5)}{\overline{p}_i^T}\right]\cdot\int dm+\\ +\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_i^0\overset{(5)}{[R]}\cdot\left[\int{}^i\overline{r}_i^*\cdot{}^i\overline{r}_i^{*T}\cdot dm+{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}_{C_i}^T\cdot\int dm\right]\cdot{}_i^0\overset{(5)}{[R]^T}\right\}=\\ \frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_i^0\overset{(5)}{[R]}\cdot\left[{}^iI_{pi}^*+M_i\cdot{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}_{C_i}^T\right]\cdot{}_i^0\overset{(5)}{[R]^T}\right\}+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(5)}{\overline{p}_i}\cdot\overset{(5)}{\overline{p}_i^T}\right]\cdot M_i\end{gathered}\right\} \tag{45}$$

The time derivatives of fifth order for the rotation matrix and position vector, included in the above equation, are determined as follows:

$$\left\{\begin{gathered}\left\{{}_i^0\overset{(5)}{[R]^{(T)}};\overset{(5)}{\overline{p}_i^{(T)}}\right\}=\frac{d}{dt}\left\{\sum_{j=1}^{i}A_{ij}^{(T)}\left[(R);(\overline{p})\right]\cdot\dddot{q}_j+4\cdot\sum_{j=1}^{i}\sum_{k=1}^{i}A_{ijk}^{(T)}\left[(R);(\overline{p})\right]\dddot{q}_j\cdot\dot{q}_k+\right.\\ \left.+3\cdot\sum_{j=1}^{i}\sum_{k=1}^{i}A_{ijk}^{(T)}\left[(R);(\overline{p})\right]\ddot{q}_j\cdot\ddot{q}_k\right\}+\frac{d}{dt}\left\{6\cdot\sum_{j=1}^{i}\sum_{k=1}^{i}\sum_{m=1}^{i}A_{ijkm}^{(T)}\left[(R);(\overline{p})\right]\cdot\ddot{q}_j\cdot\dot{q}_k\cdot\dot{q}_m+\right.\\ \left.+\sum_{j=1}^{i}\sum_{k=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i}A_{ijkmp}^{(T)}\left[(R);(\overline{p})\right]\cdot\dot{q}_j\cdot\dot{q}_k\cdot\dot{q}_m\cdot\dot{q}_p\right\}\end{gathered}\right\} \tag{46}$$

The differential matrices of various orders, included in (46), are determined in accordance with (12) – (17) and (28) – (33), on the basis of the same matrix exponentials functions and their time derivatives. The starting expression (45) corresponding to acceleration energy of fourth order and (46) corresponding to time derivatives of fifth order for the rotation matrix and position vector, will be developed in the final form in the other papers of the author.

3. Generalized and Operational Accelerations of Higher Order

The expressions of definition for acceleration energies, presented in the previous chapter, are time functions by the generalized coordinates and their time derivatives of higher order. Taking into study a mechanical robot structure (see Fig. 3), as integrating part in MBS, the position and orientation of the last moving frame $\{n+1\}$ are included in a column matrix ${}^{0}\overline{X}(t)$, having (6×1) size. Considering the researches from [3] – [4], in this paper the following expressions are proposed:

$$
\begin{aligned}
{}^{0}\overset{(m)}{\overline{X}}(t) &= \sum_{k=1}^{m}\frac{(m-1)!}{(k-1)!(m-k)!}\cdot \overset{(k-1)}{{}^{0}J\left[\overline{\theta}(t)\right]}\cdot \overset{[m-(k-1)]}{\theta(t)} = \\
&= {}^{0}J\left[\overline{\theta}(t)\right]\cdot \overset{(m)}{\theta}(t) + \sum_{k=1}^{m-1}\frac{(m-1)!}{k!(m-k-1)!}\cdot \overset{(k)}{{}^{0}J\left[\overline{\theta}(t)\right]}\cdot \overset{[m-k]}{\theta}(t),
\end{aligned} \tag{47}
$$

$$
\overset{(m)}{\theta}(t) = {}^{0}J\left[\overline{\theta}(t)\right]^{-1}\cdot {}^{0}\overset{(m)}{\overline{X}}(t) - {}^{0}J\left[\overline{\theta}(t)\right]^{-1}\cdot \sum_{k=1}^{m-1}\frac{(m-1)!}{k!(m-k-1)!}\cdot \overset{(k)}{{}^{0}J\left[\overline{\theta}(t)\right]}\cdot \overset{(m-k)}{\theta}(t),\ m\geq 1 . \tag{48}
$$

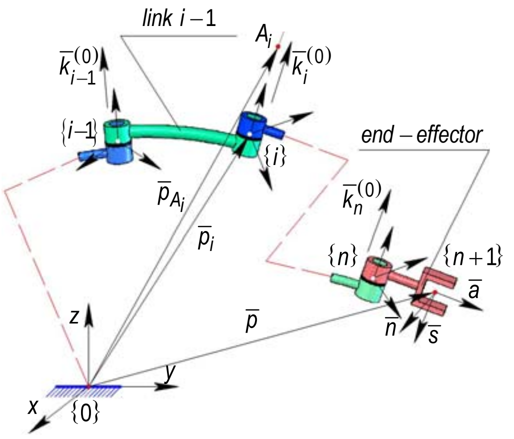


Fig. 3 Sequence from a mechanical robot structure (MRS)

In the above expressions, the symbols have the following significances: (m) is the order of the time derivatives; ${}^{0}\overset{(m)}{\overline{X}}(t)$ is the column matrix of the operational accelerations of higher order; $\overset{(m)}{\theta}(t)$ is the column matrix of generalized accelerations of higher order, and ${}^{0}J\left[\overline{\theta}(t)\right]$ is named Jacobian matrix or the velocities transfer matrix. Considering the mathematical models from [4], the Jacobian matrix can be determined with matrix exponentials. At beginning, the following matrices, as functions of exponentials, are established:

$$\underset{(3x3)}{ME}\left[V_{i1}\right]=\exp\left\{\sum_{j=0}^{i-1}\left[\overline{k}_j^{(0)}\times\right]\cdot q_j\cdot\Delta_j\right\};\ \underset{(3x6)}{ME}\left[V_{i2}\right]=\left[\begin{matrix} I_3 & \Delta_i\cdot\left[\overline{k}_i^{(0)}\times\right]\end{matrix}\right] \tag{49}$$

$$\underset{\left\{6x\left[6+3\cdot(n-i)\right]\right\}}{ME\left(V_{i3}\right)}=\left\{ME\left(V_{i31}^{*}\right)=\begin{bmatrix} I_3 \\ \left[0\right]_{3\times3}\end{bmatrix}\quad ME\left(V_{i33}^{*}\right)\quad ME\left(V_{i32}^{*}\right)\right\}, \tag{50}$$

where $ME\left(V_{i33}^{*}\right)=\begin{bmatrix} \left[0\right]_{3\times3} \\ \exp\left\{\sum_{k=i}^{n}\left\{\overline{k}_k^{(0)}\times\right\}q_k^{*}\right\} \\ where\ \ k=i\rightarrow n\end{bmatrix}$, $ME\left(V_{i32}^{*}\right)=\begin{bmatrix} \left[0\right]_{3\times3} \\ \exp\left\{\sum_{m=i-1}^{k-1}\left\{\overline{k}_m^{(0)}\times\right\}q_m^{*}\cdot\delta_m\right\}\end{bmatrix}$.

Applying a series of transformations, within (12) - (14), the following exponential expressions are obtained:

$$\underset{(6\times6)}{ME}\left\{J_{i1}\left[\theta_i\left(t\right)\right]\right\}=\begin{bmatrix} ME\left[V_{i1}\right] & \left[0\right] \\ \left[0\right] & ME\left[V_{i1}\right]\end{bmatrix};\ \underset{(6\times9)}{ME}\left\{J_{i2}\left[\theta_i\left(t\right)\right]\right\}=\begin{bmatrix} ME\left[V_{i2}\right] & \left[0\right] \\ \left[0\right] & I_3\end{bmatrix};$$

$$\underset{\{9\times[9+3\cdot(n-i)]\}}{ME}\left\{J_{i3}\left[\theta_i\left(t\right)\right]\right\}=\begin{bmatrix} ME\left[V_{i3}\right] & \left[0\right] \\ \left[0\right] & I_3\end{bmatrix}; \tag{51}$$

$$\underset{\{[9+3\cdot(n-i)]\times1\}}{M_{iv\omega}}=\left\{\left\{M_{iv}^{T}=\left[\overline{v}_i^{(0)T}\quad\left[\overline{b}_k^{T}\quad k=i\rightarrow n\right]\quad\overline{p}_n^{(0)T}\right]\right\}\quad\Delta_i\cdot\overline{k}_i^{(0)T}\right\}^{T} \tag{52}$$

$$\overline{v}_k^{(0)}=\left\{\overline{p}_k^{(0)}\times\right\}\overline{k}_k^{(0)}\cdot\Delta_k+\left(1-\Delta_k\right)\cdot\overline{k}_k^{(0)} \tag{53}$$

$$\begin{aligned}\overline{b}_k\left(t\right)=&\left\{I_3\cdot q_k\left(t\right)+\left[\overline{k}_k^{(0)}\times\right]\cdot\left\{1-c\left[q_k\left(t\right)\cdot\Delta_k\right]\right\}+\right.\\&\left.+\overline{k}_k^{(0)}\cdot\overline{k}_k^{(0)T}\cdot\left\{q_k\left(t\right)-s\left[q_k\left(t\right)\cdot\Delta_k\right]\right\}\right\}\cdot\overline{v}_k^{(0)}\end{aligned} \tag{54}$$

The expression (52) is the column vector of screw parameters (homogeneous coordinates), as well as position and orientation parameters of the robot's end-effector. They are included in the Jacobian matrix, and its time derivative of $\left(k\right)$ order [4], according to expressions below presented in a symbolical form:

$$\underset{(6\times n)}{{}^{0}J\left[\overline{\theta}\left(t\right)\right]}\equiv\left[\underset{(6\times1)}{{}^{0}J_i\left[\overline{\theta}_i\left(t\right)\right]}\quad where\ \ i=1\rightarrow n\right], \tag{55}$$

$$\underset{(6\times1)}{{}^{0}J_i\left[\overline{\theta}_i\left(t\right)\right]}=ME\left\{J_{i1}\left[\overline{\theta}_i\left(t\right)\right]\right\}\cdot ME\left\{J_{i2}\left[\overline{\theta}_i\left(t\right)\right]\right\}\cdot ME\left\{J_{i3}\left[\overline{\theta}_i\left(t\right)\right]\right\}\cdot M_{iv\omega}\left[\overline{\theta}_i\left(t\right)\right]$$

$$\underset{(6\times n)}{{}^{0}J\left[\overset{(k)}{\overline{\theta}}\left(t\right)\right]}\equiv\left[\underset{(6\times1)}{{}^{0}J_i\left[\overset{(k)}{\overline{\theta}_i}\left(t\right)\right]}\quad where\ \ i=1\rightarrow n\right], \tag{56}$$

$$ {}^{0}J_i\Big[\overset{(k)}{\overline{\theta}_i}(t)\Big]_{(6\times1)} = \begin{Bmatrix} ME\left\{J_{i1}\left[\overset{(k)}{\overline{\theta}_i}(t)\right]\right\}\cdot ME\left\{J_{i2}\left[\overline{\theta}_i(t)\right]\right\}\cdot ME\left\{J_{i3}\left[\overline{\theta}_i(t)\right]\right\}\cdot M_{iv\omega}\left[\overline{\theta}_i(t)\right] + \\ +ME\left\{J_{i1}\left[\overline{\theta}_i(t)\right]\right\}\cdot ME\left\{J_{i2}\left[\overline{\theta}_i(t)\right]\right\}\cdot ME\left\{J_{i3}\left[\overset{(k)}{\overline{\theta}_i}(t)\right]\right\}\cdot M_{iv\omega}\left[\overline{\theta}_i(t)\right] + \\ +ME\left\{J_{i1}\left[\overline{\theta}_i(t)\right]\right\}\cdot ME\left\{J_{i2}\left[\overline{\theta}_i(t)\right]\right\}\cdot ME\left\{J_{i3}\left[\overline{\theta}_i(t)\right]\right\}\cdot M_{iv\omega}\left[\overset{(k)}{\overline{\theta}_i}(t)\right] \end{Bmatrix}. $$

Therefore, using (47) and (48) as well as the polynomial interpolating functions of higher order, the time variation law of the acceleration energies is obtained for any application in the multibody systems.

4. Advanced Principles and Motion Equations in Analytical Dynamics

In the Newtonian dynamics of the multibody mechanical systems (MBS), the most general theorem is considered the kinetic energy theorem in the differential form. In keeping with [1] – [5], the advanced studies on the MBS have led to existence of some higher energy, corresponding to accelerations of higher order.

According to the literature [1], [2], [4] and [14] – [15], the general principles underlying the entire analytical dynamics are: virtual work principle, the principle of D`Alembert - Lagrange, which is known as the principle of virtual work, specific to dynamic behavior of mechanical systems.

4.1 Formulations on Advanced Differential Principles

For explain the advanced principles in analytical dynamics, at beginning a few notations and symbols are presented. So, according to Fig. 1, the position and orientation of each of the (n) bodies is expressed by:

$$ \overline{r}_{C_i}(t) = \overline{r}_{C_i}\left[q_j(t),\ j=1\to k\right]; \qquad \overline{\psi}_i(t) = \overline{\psi}_i\left[q_j(t)\cdot\Delta_j,\ j=1\to k\right] \tag{57} $$

where $\overline{r}_{C_i}$ the position vector of the mass center of each is body, and $\overline{\psi}_i$ is an angular vector specific to orientation of each body in Cartesian space. These vectors are depending of generalized coordinates q_j which unequivocally defining the motion of the entire mechanical system (MBS). It implements the observation that $\Delta_j = \{(1, for Translation); (0, for Rotation)\}$. In simplifying assumptions, MBS is a holonomic mechanical system, and therefore the two vectors are associated virtual displacements, according to differential expressions:

$$ \delta\overline{r}_{C_i} = \sum_{j=1}^{k}\frac{\partial\overline{r}_{C_i}}{\partial q_j}\cdot\delta q_j\ ; \quad \delta\overline{\psi}_i = \sum_{j=1}^{k}\frac{\partial\overline{\psi}_i}{\partial q_j}\cdot\Delta_j\cdot\delta q_j = \sum_{j=1}^{k}J_{i\psi}\cdot\delta q_j\ . \tag{58} $$

As a result, the following conditions are obvious for holonomic mechanical system:

$$ \left\{\delta q_j \neq 0,\quad \delta q_i = 0,\ where\quad i=1\to k,\ j=1\to k\right\} \tag{59} $$

In keeping with other researches of the author, [4], [5] and [11], for a multibody rigid system, symbolically presented in Fig.1, the differential principle under generalized form in analytical dynamics (generalization of D'Alembert - Lagrange principle) is defined with the next differential expression:

$$\sum_{i=1}^{n}\left(\overline{F}_i^* - M_i \cdot \overline{a}_{C_i}\right)\cdot \delta\overline{r}_{C_i} + \sum_{i=1}^{n}\left(\overline{N}_i^* - I_i^* \cdot \overline{\varepsilon}_i - \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i\right)\cdot \delta\overline{\psi}_i = 0 \qquad (60)$$

where $\overline{F}_i^*$, $\overline{N}_i^*$ are the active forces and theirs moments with respect to mass center, and I_i^* is the axial and centrifugal inertia tensor (see expression(5)). The expression (60) contains the angular parameters: $\overline{\omega}_i$ and $\overline{\varepsilon}_i$, named the angular velocity and acceleration corresponding to a resultant rotational movements of each body belonging to the system. As a result, from (60), a generalization of the Lagrange`s Equations of first kind, according to [4], [5], [14] and [15], for a multibody system is:

$$\begin{aligned}&\sum_{i=1}^{n} M_i \cdot \overline{a}_{C_i}^T \cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n}\left(I_i^* \cdot \overline{\varepsilon}_i + \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i\right)^T \cdot \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j = \\ &= \sum_{i=1}^{n} \overline{F}_i^{*T} \cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j} + \sum_{i=1}^{n} \overline{N}_i^{*T} \cdot \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j\end{aligned} \qquad (61)$$

Applying the time derivatives of first, second and then (k) order on (61) it results:

$$\begin{aligned}&\frac{d}{dt}\left[\sum_{i=1}^{n}\left(\overline{F}_i^* - M_i \cdot \overline{a}_{C_i}\right)\cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j}\right] + \frac{d}{dt}\left[\sum_{i=1}^{n}\left(\overline{N}_i^* - I_i^* \cdot \overline{\varepsilon}_i - \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i\right)\cdot \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j\right] = 0;\\ &\frac{d^2}{dt^2}\left[\sum_{i=1}^{n}\left(\overline{F}_i^* - M_i \cdot \overline{a}_{C_i}\right)\cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j}\right] + \frac{d^2}{dt^2}\left[\sum_{i=1}^{n}\left(\overline{N}_i^* - I_i^* \cdot \overline{\varepsilon}_i - \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i\right)\cdot \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j\right] = 0; \quad (62)\\ &\frac{d^k}{dt^k}\left[\sum_{i=1}^{n}\left(\overline{F}_i^* - M_i \cdot \overline{a}_{C_i}\right)\cdot \frac{\partial \overline{r}_{C_i}}{\partial q_j}\right] + \frac{d^k}{dt^k}\left[\sum_{i=1}^{n}\left(\overline{N}_i^* - I_i^* \cdot \overline{\varepsilon}_i - \overline{\omega}_i \times I_i^* \cdot \overline{\omega}_i\right)\cdot \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j\right] = 0.\end{aligned}$$

Applying important differential transformations on the position and orientation parameters (57), in keeping with [11], the partial and time derivatives are obtained:

$$\frac{\partial \overline{r}_{C_i}}{\partial q_j} = \frac{\partial \overset{(m)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q_j}} \qquad \frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j = \frac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot \Delta_j; \qquad (63)$$

$$\frac{d}{dt}\left(\frac{\partial \overline{r}_{C_i}}{\partial q_j}\right) = \frac{1}{m+1}\cdot \frac{\partial \overset{(m-1)}{\overline{a}_{C_i}}}{\partial \overset{(m)}{q_j}} = \frac{1}{m+1}\cdot \frac{\partial \overset{(m+1)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q_j}}, \qquad m = 1,2,3,\ldots \qquad (64)$$

$$\frac{d}{dt}\left(\frac{\partial \overline{\psi}_i}{\partial q_j}\cdot \Delta_j\right) = \frac{1}{m+1}\cdot \frac{\partial \overset{(m-1)}{\overline{\varepsilon}_i}}{\partial \overset{(m)}{q_j}}\cdot \Delta_j = \frac{1}{m+1}\cdot \frac{\partial \overset{(m+1)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}}\cdot \Delta_j, \qquad m = 1,2,3,\ldots \qquad (65)$$

$$\frac{d^2}{dt^2}\left(\frac{\partial \overline{r}_{C_i}}{\partial q_j}\right)=\left(\frac{1}{\sum_{k=1}^{m+1} k}\right)\cdot\frac{\partial \overset{(m+2)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q}_j}=\frac{2}{(m+1)\cdot(m+2)}\cdot\frac{\partial \overset{(m+2)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q}_j}; \tag{66}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial \overline{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\left(\frac{1}{\sum_{k=1}^{m+1} k}\right)\cdot\frac{\partial \overset{(m)}{\overline{\varepsilon}_i}}{\partial \overset{(m)}{q}_j}\cdot\Delta_j=\frac{2}{(m+1)\cdot(m+2)}\cdot\frac{1}{m+1}\cdot\frac{\partial \overset{(m+2)}{\overline{\psi}_i}}{\partial \overset{(m)}{q}_j}\cdot\Delta_j; \tag{67}$$

$$\text{where } m=0,1,2,3,\ldots \text{ and } \overset{(0)}{q}_j=q_j .$$

As a result, the above differential expressions are written under the generalized form:

$$\left\{\begin{array}{c}\dfrac{d^{k-1}}{dt^{k-1}}\left(\dfrac{\partial \overline{r}_{C_i}}{\partial q_j}\right)=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial \overset{(m+k-1)}{\overline{r}_{C_i}}}{\partial \overset{(m)}{q}_j}\\ k\ge 1;\; k=\{1;2;3;4;5;.....\};\;\; m\ge(k+1);\; m=\{2;3;4;5;.....\}\end{array}\right\}; \tag{68}$$

$$\left\{\begin{array}{c}\dfrac{d^{k-1}}{dt^{k-1}}\left(\dfrac{\partial \overline{\psi}_i}{\partial q_j}\cdot\Delta_j\right)=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial \overset{(m+k-3)}{\overline{\varepsilon}_i}}{\partial \overset{(m)}{q}_j}\cdot\Delta_j=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial \overset{(m+k-1)}{\overline{\psi}_i}}{\partial \overset{(m)}{q}_j}\cdot\Delta_j\\ k\ge 1;\; k=\{1;2;3;4;5;.....\};\;\; m\ge(k+1);\; m=\{2;3;4;5;.....\}\end{array}\right\}. \tag{69}$$

The above differential equations of position and orientation are substituted in the differential principle under the generalized form (61). So, the motion equations of higher order will be obtained for anything MBS.

4.2 The Motion Equations of Higher Order

First of all, Tsenov – Mangeron formulation in the Newtonian dynamics [2] is presented below:

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}_j}\right)-\frac{\partial E_C}{\partial q_j}=\frac{1}{m}\cdot\left(\frac{\partial \overset{(m)}{E_C}}{\partial \overset{(m)}{q}_j}-(m+1)\cdot\frac{\partial E_C}{\partial q_j}\right)=Q_j^*,\left\{\begin{array}{r}m=1,2,\ldots \text{ is time deriving order}\\ \text{and } j=1\rightarrow n\end{array}\right\} \tag{70}$$

This expression is a development on the Lagrange’s equations of second kind, but taking into account the generalized accelerations of higher order. The symbol E_C is the total kinetic energy for mechanical system with $n\ d.o.f.$ subjected to dynamical study, and Q_j^* are the generalized forces corresponding to $n\ d.o.f.$

Whereas the some mechanical systems are dominated by accelerations of higher order, especially in the case of suddenly motions, the researches of the author have led at *the generalization of Gibbs-Appell equations*:

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}_j}\right)-\frac{\partial E_C}{\partial q_j}=\frac{\partial \overset{(m-2)}{E_A^{(1)}}}{\partial \overset{(m)}{q_j}}=Q_{i\ddot{o}}^j\ ,\ \left\{\begin{array}{l} where\quad \overset{(0)}{E_A^{(1)}}=E_A^{(1)}\quad j=1\rightarrow n,\quad k=1 \\ m\geq\left[\left(k+1\right)=2\right]\ is\ time\ deriving\ operator \end{array}\right\};\ (71)$$

where $Q_{i\ddot{o}}^j$ is named the generalized inertia force. It observes that the above equations are corresponding to holonomic systems, and they are functions of generalized accelerations of higher order. In the same time, the central function in these equations is the acceleration energy of first order (4) and (6).

When the mechanical system is dominated by external forces characterized by certain variation law in relation to time, the existence of higher order energies is noticed, see [8] - [12]. So, applying a series of complex matrix and differential transformations on differential principle in the generalized form (61), the author of this paper have been proposed [12] the following development for the motion equations of higher order:

$$\left\{\begin{array}{c} \frac{1}{m+1}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[2\cdot \overset{(m-3)}{E_A^{(2)}}+\overset{(m-1)}{E_A^{(1)}}\right]=\dot{Q}_{i\ddot{o}}^j\ \left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\cdots\overset{(m)}{\overline{\theta}}(t)\right] \\ j=1\rightarrow n,\ k=2,\ m\geq\left[\left(k+1\right)=3\right],\quad \overset{(0)}{E_A^{(2)}}=E_A^{(2)} \end{array}\right\}.\quad (72)$$

Within of the above differential equations, the central functions and their time derivatives are acceleration energies of first and second order (see expressions (4) and (6), and respectively (21) and (22)). In the right member from (72) is founded the first time derivative of the generalized inertia forces, mechanically equivalent with the generalized percussion forces.

Extending the study on the acceleration energies of third and fourth orders, the author of this paper has been proposed the following development for the differential equations of motion of higher order:

$$\left\{\begin{array}{l} \frac{2}{(m+1)\cdot(m+2)}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[5\cdot \overset{(m-4)}{E_A^{(3)}}+2\cdot \overset{(m-2)}{E_A^{(2)}}+\overset{(m)}{E_A^{(1)}}\right]=\ddot{Q}_{i\ddot{o}}^j\ \left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\cdots\overset{(m)}{\overline{\theta}}(t)\right] \\ where\quad j=1\rightarrow n,\quad k=3,\ m\geq\left[\left(k+1\right)=4\right],\quad m=4,5,6,\ldots\quad and\ \overset{(0)}{E_A^{(3)}}=E_A^{(3)} \end{array}\right\};\ (73)$$

$$\left\{\begin{array}{l} \frac{2\cdot 3}{(m+1)\cdot(m+2)\cdot(m+3)}\cdot\frac{\partial}{\partial \overset{(m)}{q_j}}\left[9\cdot \overset{(m-5)}{E_A^{(4)}}+5\cdot \overset{(m-3)}{E_A^{(3)}}+2\cdot \overset{(m-1)}{E_A^{(2)}}+\overset{(m+1)}{E_A^{(1)}}\right]= \\ \qquad =\dddot{Q}_{i\ddot{o}}^j\ \left[\overline{\theta}(t);\dot{\overline{\theta}}(t);\ddot{\overline{\theta}}(t);\cdots;\overset{(m)}{\overline{\theta}}(t)\right] \\ where\quad j=1\rightarrow n,\quad k=4,\quad m\geq\left[\left(k+1\right)=5\right],\quad m=5,6,7,\ldots\ and\ \overset{(0)}{E_A^{(4)}}=E_A^{(4)} \end{array}\right\};\quad (74)$$

So, unlike the equations (73) in which the acceleration energies of first, second and third order are founded, in the (74) is also included the acceleration energy of fourth order as well as their time derivatives (see (4) and (6), (21) and (22), (38) and (40), respectively (45)). In the right member from (73) and (74) is founded the second and respectively the third time derivatives of the generalized inertia forces.

Any mechanical system founded in the dynamical behavior is dominated on the one hand through the generalized inertia forces, on the other hand by generalized gravitational forces. Using the researches of the author form [4] and [11], the expression of definition and the time derivatives are shown below:

$$Q_g^i = \sum_{j=i}^{n} M_j \cdot \left\{ \left\{ {}_i^0[R]^T \cdot \overline{g} \right\}^T \cdot \left(1-\Delta_i\right) + \left\{ {}_i^0[R]^T \cdot \left[\left({}^0\overline{r}_{C_j} - \overline{p}_i \right) \times \overline{g} \right] \right\}^T \cdot \Delta_i \right\} \cdot {}^i\overline{k}_i \, ; \quad (75)$$

$$\left\{ \begin{array}{c} \dot{Q}_g^j = \sum_{i=1}^{n} M_i \cdot \overline{g}^T \cdot \left[\dfrac{1}{m+1} \cdot \dfrac{\partial \overset{(m+1)}{\overline{r}_{Ci}}}{\partial \overset{(m)}{q_j}} \right] + \sum_{i=1}^{n} \dot{\overline{r}}_{C_i} \times M_i \cdot \overline{g}^T \cdot \left[\dfrac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \right] + \\ + \sum_{i=1}^{n} \overline{r}_{C_i} \times M_i \cdot \overline{g}^T \cdot \left[\dfrac{1}{m+1} \cdot \dfrac{\partial \overset{(m+1)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \right] \\ where \quad m = 3,4,5,\ldots \; \overline{g} = \tau \cdot g \cdot \overline{k}_0 \\ \tau = \mp \overline{k}_0^T \cdot \overline{k}_g, \; \overline{k}_g = {}^0\overline{g} / \left| {}^0\overline{g} \right|, \; \overline{k}_0 - vertical\ unit\ vector \in \{0\} \end{array} \right\} ; \quad (76)$$

$$\left\{ \begin{array}{c} \ddot{Q}_g^j(t) = \sum_{i=1}^{n} M_i \cdot \overline{g}^T \cdot \left[\dfrac{2}{(m+1)\cdot(m+2)} \cdot \dfrac{\partial \overset{(m+2)}{\overline{r}_{Ci}}}{\partial \overset{(m)}{q_j}} \right] + \sum_{i=1}^{n} \ddot{\overline{r}}_{C_i} \times M_i \cdot \overline{g}^T \cdot \left[\dfrac{\partial \overset{(m)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \right] + \\ +2 \cdot \sum_{i=1}^{n} \dot{\overline{r}}_{C_i} \times M_i \cdot \overline{g}^T \cdot \left[\dfrac{1}{m+1} \cdot \dfrac{\partial \overset{(m+1)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \right] + \\ + \sum_{i=1}^{n} \overline{r}_{C_i} \times M_i \cdot \overline{g}^T \cdot \left[\dfrac{2}{(m+1)\cdot(m+2)} \cdot \dfrac{\partial \overset{(m+2)}{\overline{\psi}_i}}{\partial \overset{(m)}{q_j}} \cdot \Delta_j \right] \\ where \quad m = 4,5,6,\ldots \; \overline{g} = \tau \cdot g \cdot \overline{k}_0, \; \tau = \mp \overline{k}_0^T \cdot \overline{k}_g \\ \overline{k}_g = {}^0\overline{g} / \left| {}^0\overline{g} \right|, \; \overline{k}_0 - vertical\ unit\ vector \in \{0\} \end{array} \right\} . \quad (77)$$

The above expressions characterize the generalized gravitational forces and their time derivatives. The significances of the symbols from the above expressions have been shown in first chapter of this paper.

On the basis of the differential transformations (2), (62), (68) and (69), as well as the differential motion equations (71) – (74), the author of this paper proposes *the generalized differential equations of higher order* in the case of the mechanical systems (MBS), dynamically characterized by suddenly and transitory motions:

$$\left\{\begin{array}{c} \dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{q_j}}\left\{\sum_{p=1}^{k}\left[\dfrac{p\cdot(p+1)}{2}-\delta_p\right]\cdot E_A^{(p)}\overset{(m+k)-(2\cdot p+1)}{\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(p+1)}{\bar{\theta}}(t)\right]}\right\}= \\ =\overset{(k-1)}{Q^j_{i\ddot{o}}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] \\ k\geq 1;\ \ k=\{1;2;3;4;5;.....\};\ m\geq(k+1);\ \ m=\{2;3;4;5;.....\} \\ p=1\rightarrow k\ ;\ \ \delta_p=\{\{0;p=1\};\{1;p>1\}\} \end{array}\right\}. \quad (78)$$

Using (75) – (77), finally, the generalized gravitational forces and their higher order derivatives are determined with the following:

$$\left\{\begin{array}{c} \overset{(k-1)}{Q^j_g}(t)=\dfrac{(k-1)!\cdot m!}{(m+k-1)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{q_j}}\sum_{i=1}^{n}\left[M_i\cdot\bar{g}^T\cdot\overset{(m+k-1)}{\bar{r}_{C_i}}+\bar{r}_{C_i}\times M_i\cdot\bar{g}^T\cdot\overset{(m+k-1)}{\bar{\psi}_i}\cdot\Delta_j\right]+ \\ +\sum_{i=1}^{n}\overset{(k-1)}{\bar{r}_{C_i}}\times M_i\cdot\bar{g}^T\cdot\left[\dfrac{\partial\overset{(m)}{\bar{\psi}_i}}{\partial\overset{(m)}{q_j}}\cdot\Delta_j\right]+ \\ +\delta_{kk}\cdot\left\{(k-1)\cdot\dfrac{(k-2)!\cdot m!}{(m+k-2)!}\cdot\dfrac{\partial}{\partial \overset{(m)}{q_j}}\sum_{i=1}^{n}\left[\overset{(k-2)}{\bar{r}_{C_i}}\times M_i\cdot\bar{g}^T\cdot\overset{(m+k-2)}{\bar{\psi}_i}\cdot\Delta_j\right]\right\} \\ \text{where }\ k\geq 1;\ \ k=\{1;2;3;4;5;.....\};\ \ \ m\geq(k+1);\ \ \ m=\{2;3;4;5;.....\} \\ p=1\rightarrow k;\ \delta_{kk}=\left[(0,\text{for }k\leq 2);\ (1,\text{for }k>2)\right];\bar{g}=\tau\cdot g\cdot\bar{k}_0,\ \ \tau=\mp\bar{k}_0^T\cdot\bar{k}_g \\ \bar{k}_g={}^0\bar{g}\,/\left|{}^0\bar{g}\right|,\ \ \bar{k}_0-\text{vertical unit vector}\in\{0\} \end{array}\right\}. \quad (79)$$

Summing the generalized inertia and the gravitational forces, according to [4], it results *the generalized driving forces*, this operation being valuable in the case of the second, third and higher order equations of motion:

$$\left\{\begin{array}{c} \overset{(k-1)}{Q^j_{i\ddot{o}}}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]+\overset{(k-1)}{Q^j_g}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right]= \\ =\overset{(k-1)}{Q^j_m}\left[\bar{\theta}(t);\dot{\bar{\theta}}(t);\cdots;\overset{(m)}{\bar{\theta}}(t)\right] \\ k\geq 1;\ \ k=\{1;2;3;4;5;.....\};\ \ \ m\geq(k+1);\ \ \ m=\{2;3;4;5;.....\} \end{array}\right\}. \quad (80)$$

It can be seen that formulations (78), (79) and (80) are systems of *n* differential equations of higher order, which will lead to the establishment of the generalized driving forces (dynamic control functions) of first, second, third and higher order which govern MBS’s suddenly motion, as well transitory motion phases.

5. Applications

In order to illustrate in an experimental form the validity of the above presented expressions, regarding the higher order acceleration energies, as well as the differential equations of higher order it was considered the rotation motion on the angular interval $(0,\ \pi)$ of the arm of a serial robot of type Fanuc LR Mate 100 iB (see Fig. 4). This serial robot structure is mechanically characterized through five degrees of freedom. Among of these, the last three have been included in this experiment into robot arm. By using a mono-axial accelerometer, it has been experimentally established the time variation law of the tangential component of the acceleration of a point belonging to the robot arm, also named characteristic point (usually it represents the extremity of the end effector), according to [4].

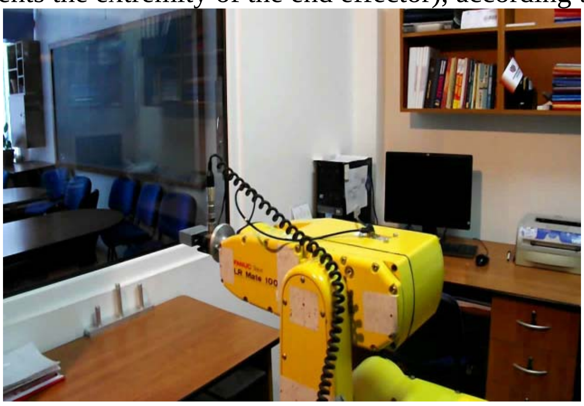


Fig.4 Fanuc LR Mate 100iB Robot (The transducer location of the robot arm)

Considering the rotation motion of the robot arm, it results in a graphical form the time variation law for the angular acceleration $\ddot{q}_3(\tau)$ (according to Fig. 5).

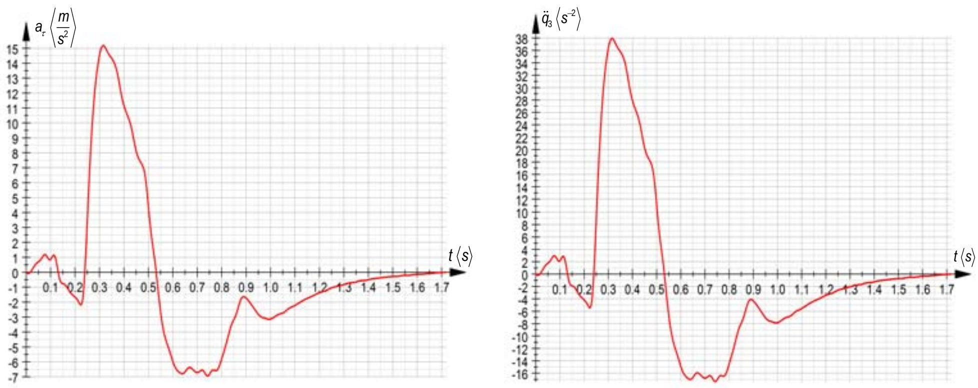


Fig. 5 The time variation of the tangential and generalized accelerations

In order to highlight the time variation law of the acceleration energies of higher order, the polynomial interpolating functions of fifth order have been applied in a new formulation, according to [10] –[12]:

$$\ddddot{q}_{ji}(\tau) = \frac{\tau_i - \tau}{t_i} \cdot \ddddot{q}_{ji}(\tau_{i-1}) + \frac{\tau - \tau_{i-1}}{t_i} \cdot \ddddot{q}_{ji}(\tau_i) \quad (81)$$

$$\dddot{q}_{ji}(\tau) = -\frac{(\tau_i - \tau)^2}{2 \cdot t_i} \cdot \ddddot{q}_{ji-1} + \frac{(\tau - \tau_{i-1})^2}{2 \cdot t_i} \cdot \ddddot{q}_{ji} + a_{ji1} \quad (82)$$

$$\ddot{q}_{ji}(\tau) = \frac{(\tau_i - \tau)^3}{6 \cdot t_i} \cdot \ddddot{q}_{ji-1} + \frac{(\tau - \tau_{i-1})^3}{6 \cdot t_i} \cdot \ddddot{q}_{ji} + a_{ji1} \cdot \tau + a_{ji2} \quad (83)$$

$$\dot{q}_{ji}(\tau) = -\frac{(\tau_i - \tau)^4}{24 \cdot t_i} \cdot \ddddot{q}_{ji-1} + \frac{(\tau - \tau_{i-1})^4}{24 \cdot t_i} \cdot \ddddot{q}_{ji} + a_{ji1} \cdot \frac{\tau^2}{2} + a_{ji2} \cdot \tau + a_{ji3} \quad (84)$$

$$q_{ji}(\tau) = \frac{(\tau_i - \tau)^5}{120 \cdot t_i} \cdot \ddddot{q}_{ji-1} + \frac{(\tau - \tau_{i-1})^5}{120 \cdot t_i} \cdot \ddddot{q}_{ji} + a_{ji1} \cdot \frac{\tau^3}{6} + a_{ji2} \cdot \frac{\tau^2}{2} + a_{ji3} \cdot \tau + a_{ji4} \quad (85)$$

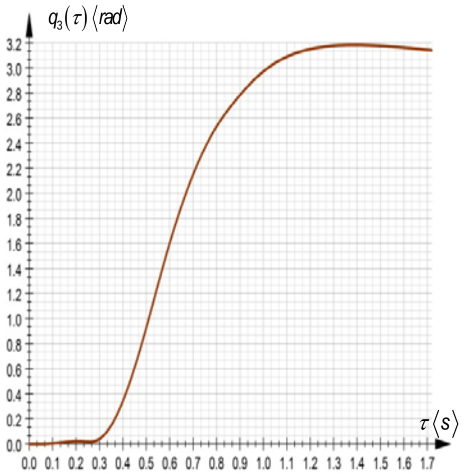


Fig. 6 Time variation law of angular coordinate

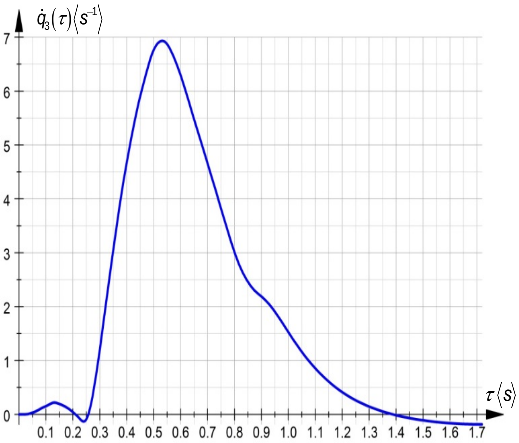


Fig. 7 Time variation law of angular velocity

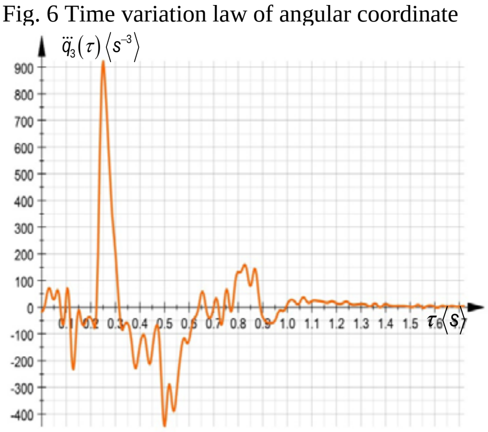


Fig. 8 Time variation law of angular acceleration of second order

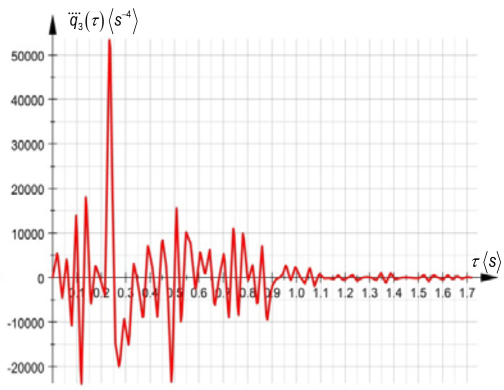


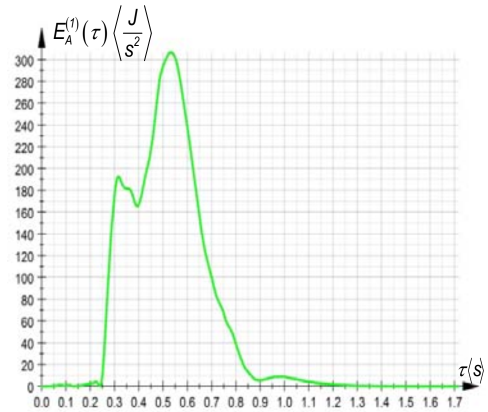
Fig. 9 Time variation law of angular acceleration of third order

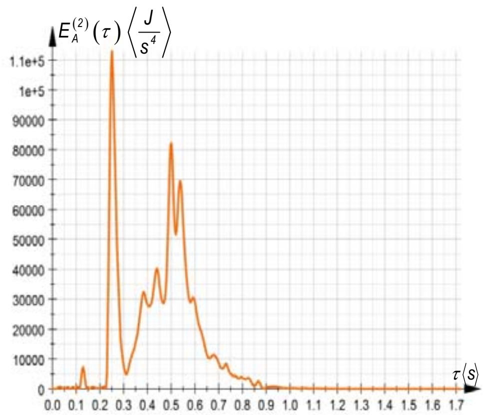
where a_{jip}, $p=1\to 4$, are the integration constants, which they are determined from the geometrical and kinematical constraints with an important role in ensuring the continuity of the rotation motion on the angular interval $(0,\ \pi)$, characterized by 51 interpolation segments. On the basis of the results from Fig. 5 and the above polynomial interpolating functions, the time variation laws for angular coordinate, velocity and acceleration of second and third order are obtained (see Fig. 6 – Fig. 9). The results, above obtained, regarding the higher order polynomial interpolating functions have been included in the expressions of the acceleration energies of first, second and third orders, corresponding to rotation motion of the robot arm:

$$E_{Aik}^{(1)}(\tau)=\frac{1}{2}\cdot\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\left(\ddot{q}_{3ik}^2(\tau)+\dot{q}_{3ik}^4(\tau)\right); \tag{86}$$

$$E_{Aik}^{(2)}(\tau)=\frac{1}{2}\cdot\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\left(\dddot{q}_{3ik}^2(\tau)-2\cdot\dot{q}_{3ik}^3(\tau)\cdot\dddot{q}_{3ik}(\tau)+9\cdot\dot{q}_{3ik}^2(\tau)\cdot\ddot{q}_{3ik}^2(\tau)+\dot{q}_{3ik}^6(\tau)\right) \tag{87}$$

$$\left\{\begin{array}{c} E_{Aik}^{(3)}(\tau)=\frac{1}{2}\Big[\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\Big(\dot{q}_{3ik}^8(\tau)-8\cdot\dot{q}_{3ik}^5(\tau)\cdot\dddot{q}_{3ik(\tau)}+30\cdot\dot{q}_{3ik}^4(\tau)\cdot\ddot{q}_{3ik}^2(\tau)- \\ -12\cdot\dot{q}_{3ik}^2(\tau)\cdot\ddot{q}_{3ik}(\tau)\cdot\ddddot{q}_{3ik}(\tau)+30\cdot\dot{q}_{3ik}^4(\tau)\cdot\ddot{q}_{3ik}^2(\tau)-12\cdot\dot{q}_{3ik}^2(\tau)\cdot\ddot{q}_{3ik}(\tau)\cdot\ddddot{q}_{3ik}(\tau)+ \\ +16\cdot\dot{q}_{3ik}^2(\tau)\cdot\dddot{q}_{3ik}^2(\tau)+24\cdot\dot{q}_{3ik}(\tau)\cdot\ddot{q}_{3ik}^2(\tau)\cdot\dddot{q}_{3ik}(\tau)+9\cdot\ddot{q}_{3ik}^4(\tau)+\ddddot{q}_{3ik}^2(\tau)\Big)\Big] \end{array}\right\} \tag{88}$$





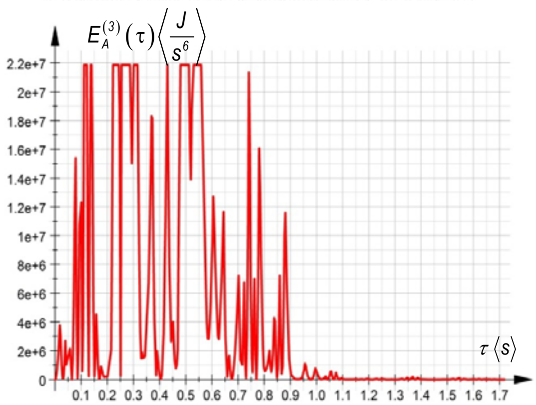


Fig.10 Time variation law of the higher acceleration energy
(a. of first order, b. of second order,
c. of third order)

To highlight the importance of expressions concerning the differential equations of motion of higher order, it was taken in study the same serial robot Fanuc of type LR Mate100iB presented in Fig.5. On the third joint of robot was induced a rotational motion in the range $\left[0,\pi\right]$. According to (71) - (80), there have been established the differential equations of motion of second, third and fourth order as follows:

$$Q^3_{mik}\left(\tau\right)=\frac{\partial E^{(1)}_{Aik}\left(\tau\right)}{\partial \ddot{q}_{3ik}\left(\tau\right)}+Q^3_{gik}\left(\tau\right)=\left(M_3\cdot x^2_{C_3}+M_3\cdot z^2_{C_3}+{}^3I_y\right)\cdot\ddot{q}_{3ik}\left(\tau\right) \quad (89)$$

$$\dot{Q}^3_{mik}\left(\tau\right)=\frac{1}{4}\cdot\frac{\partial}{\partial\dddot{q}_{3ik}\left(\tau\right)}\left[2\cdot E^{(2)}_{Aik}\left(\tau\right)+\overset{(2)}{E}{}^{(1)}_{Aik}\left(\tau\right)\right]+\dot{Q}^3_{gik}\left(\tau\right)=\left(M_3\cdot x^2_{C_3}+M_3\cdot z^2_{C_3}+{}^3I_y\right)\cdot\dddot{q}_{3ik}\left(\tau\right) \quad (90)$$

$$\left\{\begin{aligned}&\ddot{Q}^3_{mik}\left(\tau\right)=\frac{1}{15}\cdot\frac{\partial}{\partial\ddddot{q}_{3ik}\left(\tau\right)}\left[5\cdot E^{(3)}_{Aik}\left(\tau\right)+2\cdot\overset{(2)}{E}{}^{(2)}_{Aik}\left(\tau\right)+\overset{(4)}{E}{}^{(1)}_{Aik}\left(\tau\right)\right]+\ddot{Q}^3_{gik}\left(\tau\right)=\\&=\left(M_3\cdot x^2_{C_3}+M_3\cdot z^2_{C_3}+{}^3I_y\right)\cdot\ddddot{q}_{3ik}\left(\tau\right)\end{aligned}\right\} \quad (91)$$

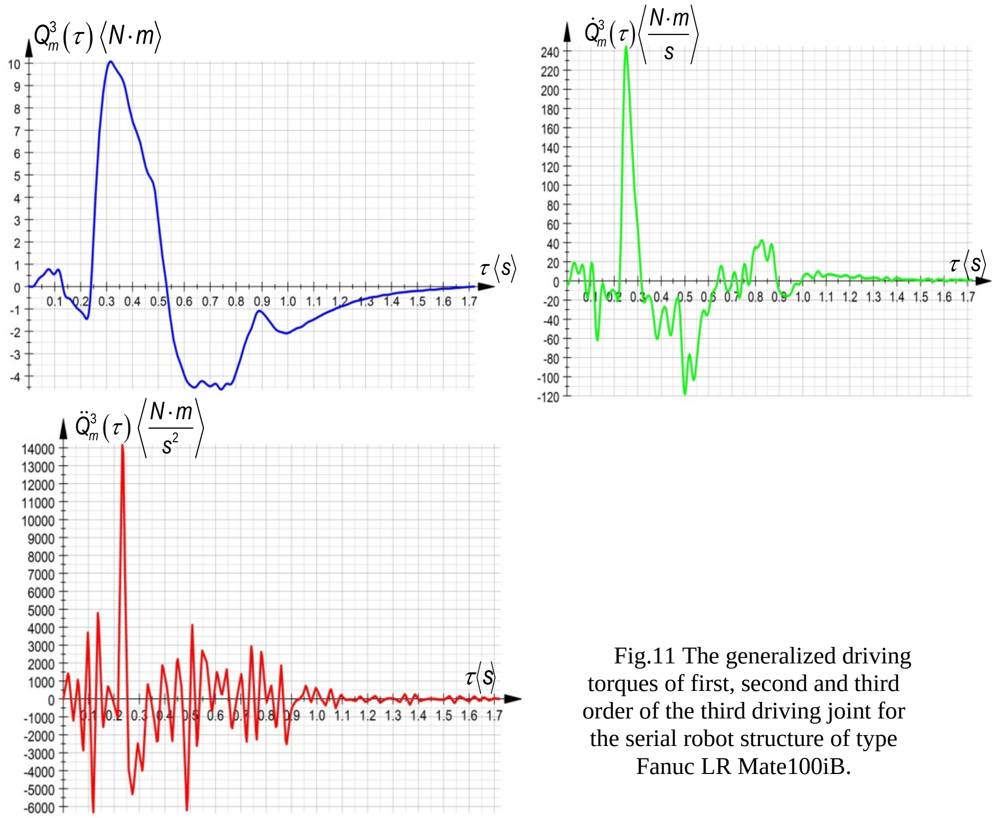


Fig.11 The generalized driving torques of first, second and third order of the third driving joint for the serial robot structure of type Fanuc LR Mate100iB.

where Q^3_{mik}, $\dot{Q}^3_{mik}$ and $\ddot{Q}^3_{mik}$ are driving torques of first, second and third order, M_3 is the mass of the robot arm, x_{C_3}, z_{C_3} are mass center coordinates and ${}^3I_y$ is the axial centrifugal inertia moment, and $\overset{(m)}{q_3}$, *where* $m = 2,3,4$ represents the angular accelerations of first, second and third order, corresponding to rotation motion for each segment of the motion trajectory in angular range $[0, \pi]$. The term $\ddot{q}_3(\tau)$ was determined by experimental measurements (see Fig.5), while $\dddot{q}_{3ik}(\tau)$ and $\ddddot{q}_{3ik}(\tau)$ were obtained by applying polynomial interpolation functions of degree five (see Fig.8 and Fig.9). On the basis of previous expressions (see (89), (90) and (91)), in Fig.11 there are represented the time variation laws for the generalized driving torques of first, second or third order, corresponding to the third driving joint of the serial robot structure of type Fanuc LR Mate100iB.

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