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Precise estimation of the resonant frequencies of mechanical structures involving a pseudo-sinc based technique

GILLICH GILBERT-RAINER*, MINDA ANDREA AMALIA, KORKA ZOLTAN-IOSIF

Universitatea "Eftimie Murgu" din Resita, P-ta Traian Vuia 1-4, 320085 Resita, Romania

Abstract. This paper presents a technique based on the pseudo-sinc function, defined by the authors, that is used to evaluate resonant frequencies with high precision as required in many engineering applications. Standard evaluation methods used to find the real frequency fail because the result depends on the acquisition time, which defines the position of the spectral lines. Commonly, interpolation, involving the amplitude peaks displayed on several spectral lines located around the maximizer, is employed to improve the frequency readability. The results can be improved in this way, but the achieved precision still depends on the acquisition time. This paper discusses the reason for leakage and proposes a new interpolation technique that is made for several maximizers attained by iteratively truncating the original signal. It was demonstrated that the maximizer obtained in this way fit on a pseudo-sinc function that is not symmetric, as assumed in the actual interpolation methods.

Keywords. Frequency estimation, leakage, resonant frequency, interpolation, polynomial function, sinc function, mechanical structure.

1. Introduction

Precise evaluation of the resonant frequencies is important in many engineering applications for identifying the structures' mechanical parameters [1] or changes of those parameters, used for instance to detect the occurrence of cracks in beams or plates [2]-[5]. Because the resonant frequencies still are the features most easily found from measurements, those shifts due to damage are qualified for use in structural health monitoring, even if they are considered to have low sensitivity to damage [2]. This supposition is made, because the frequencies evaluated with standard methods, as in all predefined frequency evaluation applications, are

* Correspondence address: gr.gillich@uem.ro

indicated on lines whose position in the spectrum is determined by the acquisition time. This happens because the frequency resolution, i.e. the distance between two consecutive spectral lines is the inverse of the signal length. Thus, a frequency shift is observed if the effect of the perturbation is higher as the half of the frequency resolution [6]-[7]. Since the acquisition time is often one second or lower, the frequency resolution is bigger than 1Hz, and therefore, the shift is observed just if it exceeds 0.5Hz, as shown in [8]. Accordingly, to get precise results, advanced evaluation techniques are requested. The signal is processed after it was acquired; the most common approach is to apply an interpolation algorithm.

The simplest actual attempts, made to improve the frequency readability, are based on interpolation techniques which always consider the maximizer (the spectral line that has associated the highest amplitude in the frequency range of interest) and one or two neighbors [9]-[14]. It is worth mentioning that the employed frequency-amplitude pairs always belong to the same spectrum. Results achieved by such interpolation get increased precision, but further depend on the acquisition time or signal length. The reason of failure is detailed in next section.

The study presented herein introduces a technique based on evaluating the frequency from one overlapped spectrum that contains maximizers form different spectra, each of them achieved for a different analysis time length. In our prior research, the interpolation was done considering that these points belong to a second-order polynomial. In this case, the evaluated frequencies are not signal-length sensitive and errors were very small compared to other methods. Later, we found out that the maximizers are distributed in accordance to a pseudo-sinc function, which is not symmetrical with the real frequency. Herein, we present this pseudo-sinc function and justify why it should be used instead of the ordinary sinc function.

2. Errors occurred in standard frequency evaluation due to leakage

We first conducted researches to establish the precision of a standard frequency evaluation method. To this aim, we consider a harmonic signal with known angular frequency ω and amplitude a. This continuous signal $x(t) = \cos(\omega t)$ is represented in the discrete form as [15]:

$$\{x\} = \{x[0], x[1], ..., x[k], ..., x[N-1]\} \tag{1}$$

In this relation, the individual elements of the sequence are:

$$x[k] = \sum_{j=0}^{N-1} a_j e^{2\pi i \frac{k}{N-1} j} \tag{2}$$

The power of the function cos is leaked out from its real frequency component into the components of the Fourier series representation. The real coefficients for a signal having the length in time domain T_S result as:

$$a_j = \frac{2}{T}\int_0^{T_S} \cos(\omega t)\cos(j\omega_0 t)dt \quad (3)$$

hence:

$$a_j = \frac{1}{T_S}\left[\frac{\sin(\omega - j\omega_0)T_S}{(\omega - j\omega_0)} + \frac{\sin(\omega + j\omega_0)T_S}{(\omega + j\omega_0)}\right] \quad (4)$$

Here, j is the spectral line number and ω_0 the angular frequency resolution.
If ω is large enough, the second term in Eq. (4) can be neglected. Taking this into consideration and substituting ω with $2\pi f$, Eq. (4) can be expressed in terms of frequency as:

$$a_j = \frac{\sin 2\pi(f_m - j\Delta f)T_S}{2\pi(f_m - j\Delta f)T_S} = \text{sinc}(2\pi(f_m - j\Delta f)T_S) \quad (5)$$

In this relation, Δf is the frequency resolution (the distance between two consecutive spectral lines), and f_m is the true frequency of the generated signal. In real applications, f_m has the meaning of the measured frequency.
The closer $j\Delta f$ approaches to the value of f_m, $f_m - j\Delta f \to 0$ and so, the sinc function approaches to value 1. Consequently, a_j approaches the true amplitude a. The more $j\Delta f$ moves away from the value of f_m, the denominator in the sinc function becomes larger. So, the coefficient a_j, given by the sinc function, becomes smaller.

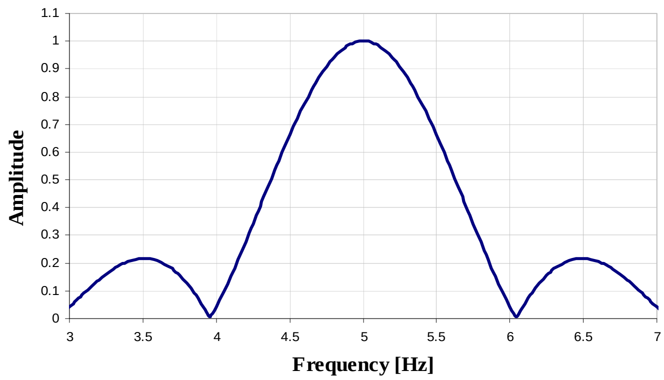

Fig. 1. The spectrum indicating the amplitudes for the sinc function.

Analyzing the amplitudes graph shown in Fig. 1, we notice that for the signal having the frequency of 5Hz, acquired for T_S = 1s the true amplitude $a = 1$ is indicated on the spectral line $j = 5$. Accurate results are obtained because

$f_m - j\Delta f = 0$. If the signal is lengthened or shortened, leakage occurs and the spectral line presumed to indicate the frequency will be removed. In this way, incorrect frequency is indicated; also the amplitude is wrongly calculated, as a_j is displayed in the spectrum.

3. Current interpolation methods and the reason for their failure

As shown in [16], to determine the real frequency, it is necessary to find a curve that performs the interpolation between three points with an amplitude peak of A_{j-1}, A_j and A_{j+1}. The central amplitude, attaining biggest value among the three points, is denoted as maximizer.
To find the real frequency, we must first determine A_{max} that is the maximum of the curve containing these points obtained from the Discrete Fourier Transform (DFT). Afterwards, the corrected frequency f_{corr} is determined by adjusting the measured frequency with a fractional correction term δ.

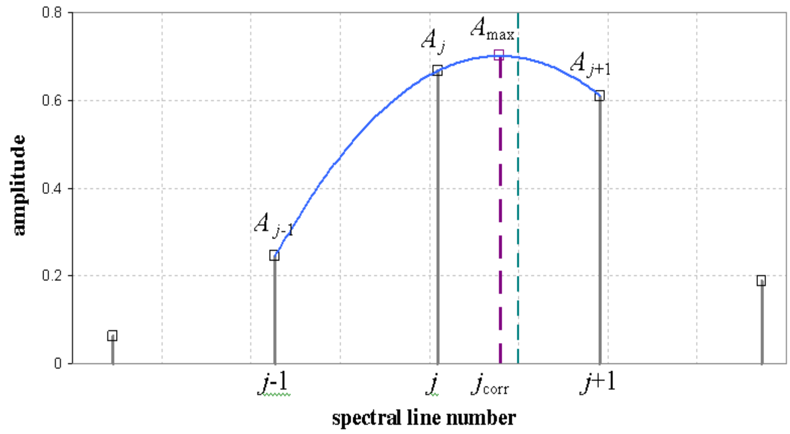

Fig. 2. The three points in the spectrum used for interpolation.

The formulas with which the correction term δ and the corrected frequencies are calculated, for the interpolation algorithms using two points, are given by Grandke [9], Quinn [10] and Jain et al.[11]. On the other hand, Ding et al [12], Voglewede [13], Jacobsen and Kootsookos [14] involve algorithms using three points of the spectrum. A comprehensive discussion regarding the achieved precision by each method is made in [17], based on a generated signal with the frequency 5 Hz and an acquisition time of around 1 second.

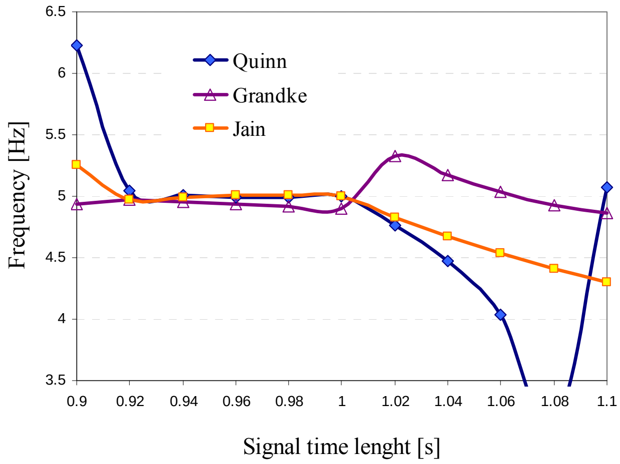

Fig. 3. Accuracy of interpolation methods that consider two points in the DFT [17].

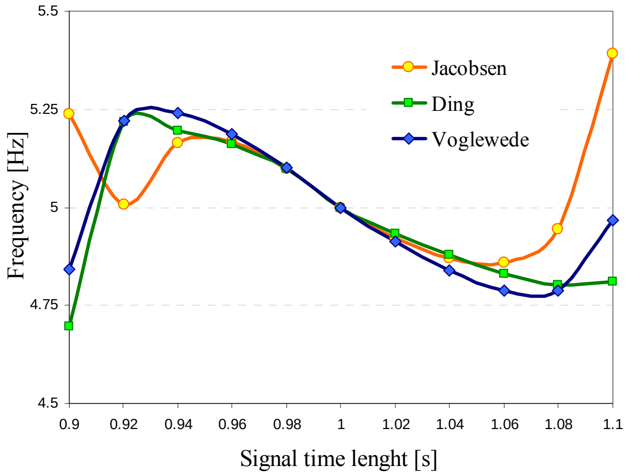

Fig. 4. Accuracy of interpolation methods that consider three points in the DFT [17].

Analyzing the accuracy of these interpolation methods, it is found that, when the signal contains an integer number of cycles, so when $T_S = 1$s, the correct frequency is obtained. If this is not the case, the standard frequency estimation method fails in indicating the true frequency [17]. If the time varies between 0.9 s and 1.1 s, a ±10% deviation is obtained for the methods considering one maximizer neighbor and ±5%, if two neighbors of the maximizer are used [18].

The accuracy of these methods is not sufficient to solve mechanical engineering problems that require high precision. For this reason, we developed a method which iteratively crops the signal until a maximum is clearly attained at the spectral line of interest [19]-[22]. This indicates that the signal contains an entire number of cycles for the studied frequency component and the measured frequency is compatible with the frequency resolution.

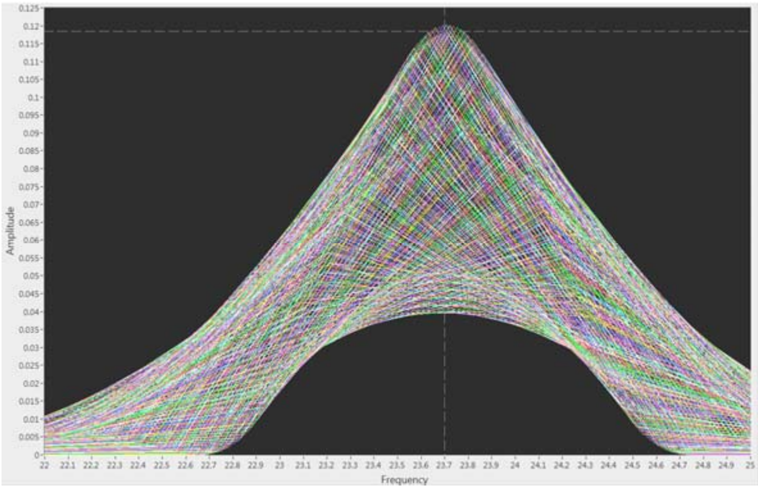

Fig. 5. Overlapped-spectrum achieved for numerous analysis times [22].

A typical representation of the overlapped spectrum containing numerous maximizers is presented in Fig. 5. Each of the maximizers shown in this figure belong to an individual spectra obtained for a particular analysis time. By identifying the highest amplitude, one can read the frequency associated to it, which is considered (and extremely close to) the true one. This method is fairly precise, errors being less than 0.5%.

4. The pseudo-sinc function proposed for an advanced estimation technique

For damage detection methods even more precise frequency estimation is requested. To this end, we studied the sinc function by comparing it with maximizers achieved from spectra obtained for different signal time lengths; this comparison is presented in Fig. 6. In this figure, one can observe a discrepancy between the sinc function (which is generally agreed to correctly indicate the maximizer for different signal time lengths) and the maximizers obtained by simulation for a generated signal with known frequency. This happens because, by changing the analysis time length T_S the frequency resolution $\Delta f=1/T_S$ also changes, so when the signal is truncated and the overlapped spectrum is obtained, the spectral lines are not equidistant [16].

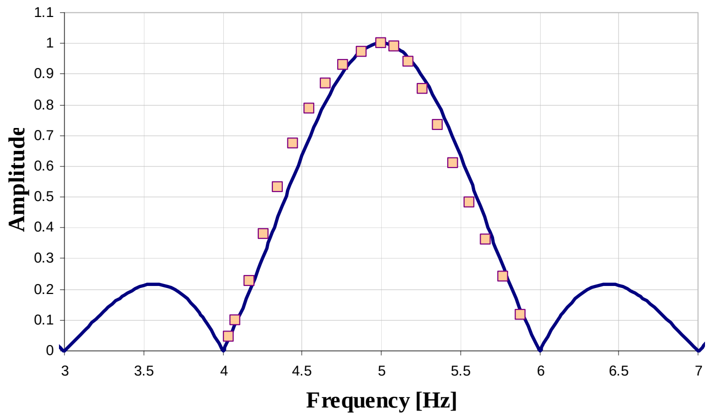

Fig. 6. Comparison between the sinc function (continuous line) and maximizers achieved by simulation (square-shaped points).

For a longer analysis time, the differences observed in Fig. 6 are no longer noticed, thus the errors achieved by frequency estimation are small and so the sinc function can be successfully used [17]-[20]. This is not the case of signals acquired for damage detection, because the amplitude of vibration signals achieved from free vibrating structures decrease rapidly, especially for the higher modes. As a consequence, the sinc function cannot be used for an accurate frequency evaluation [21]-[22]. Instead of it, we propose the use of a pseudo-sinc function which better fit the reality.

We performed experiments to find out if the pseudo-sinc function maintain its shape and fit the maximizers for any frequency and signal length. The generated signals, employed in the analysis, have following parameters: f the true frequency, a the true amplitude, T_S the analysis time of the original signal which is generated using N_S samples, n the number of cycles of length $T = 1/f$ contained in the original signals. The true amplitude a is always set as 1 and four signals, denoted with indices $h = 1...4$, and having frequencies 5Hz, 7Hz, 10Hz and 14Hz are analyzed. Three analysis time lengths are derived for each signal k considering 10, 15 respectively 20 cycles of period T. All these signals contain initially $N_S = 15000$ samples resulting different sampling rates r. A comprehensive overview regarding the involved parameters is presented in Table 1.

The four original signals are shortened and lengthened 12 times, by subtracting respectively adding 50 samples, iteratively. As a result, we obtain graphs as this presented in Fig. 7. One can observe that the amplitude a, in fact the coefficient a_j in Eq. (5), decreases if the spectral lines shift from that indicating the true frequency. In addition, it clearly results that increasing the time length $T_S = nT$ the curve becomes narrower.

Table 1. Parameters set for generating the original sinusoids to be analyzed

h	f_i [Hz]	*T* [s]	*n*	*Ts* [s]	*Ns* [-]	*r* [Hz]
1	5	0.2	20	4	15000	3750
			15	3		5000
			10	2		7500
2	7	0.142857143	20	2.857143	15000	5250
			15	2.142857		7000
			10	1.428571		10500
3	10	0.1	20	2	15000	7500
			15	1.5		10000
			10	1		15000
4	14	0.071428571	20	1.428571429	15000	10500
			15	1.071428571		14000
			10	0.714285714		21000

By performing a regression analysis for the points attained by applying the above presented procedure, we noticed that the best fit is achieved for a 5th order polynomial curve, of the form:

$$y = C_5x^5 + C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 \quad (6)$$

The regression curves for this order cross all points in the graph, proved by the coefficient of determination $R^2 = 1$, as shown in Fig. 7 for the case $h = 1$. For this signal, having the frequency $f_1 = 5\,\text{Hz}$, we obtained different curves for different number of cycles. The achieved coefficients are presented in Table 2.

Table 2. Coefficients of interpolation curves for the signal with frequency 5Hz

n	C_5	C_4	C_3	C_2	C_1	C_0
20	-137.94	3616.1	-37826	197381	-513814	533852
15	-46.867	1229.4	-12865	67137	-174725	181423
10	-9.6553	253.17	-2647	13792	-35806	37051

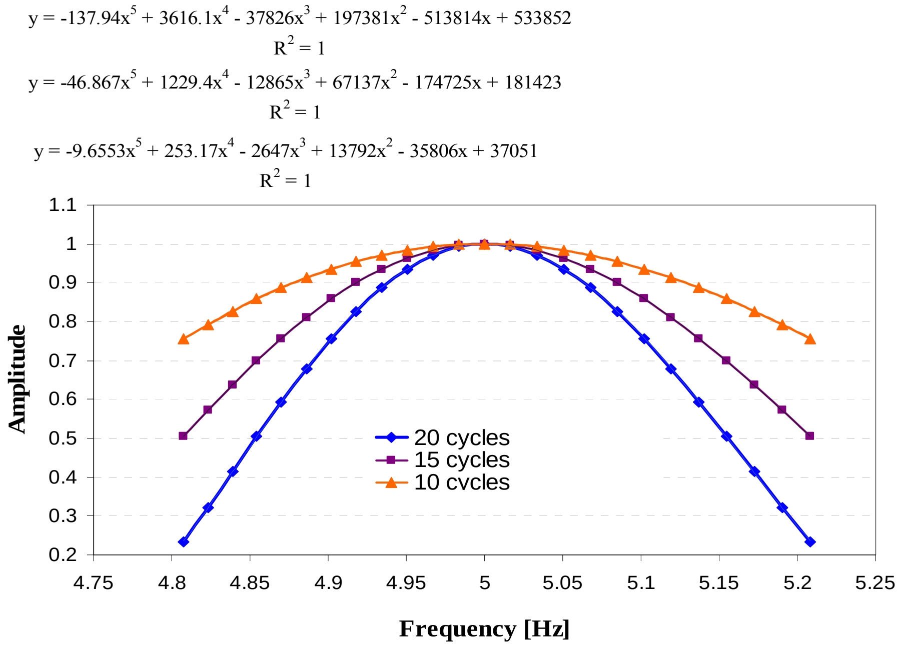

Fig. 7. The interpolation curves for f=5Hz

Then, we normalized the coefficients of the interpolation polynomials by dividing each of them to the first coefficient:

$$c_5 = C_5 / C_5;\ c_4 = C_4 / C_5;\ c_3 = C_3 / C_5;\ c_2 = C_2 / C_5;\ c_1 = C_1 / C_5;\ c_0 = C_0 / C_5 \qquad (7)$$

or in the general form:

$$c_m = C_m / C_5 \quad m = 0...5 \qquad (8)$$

For the analyzed case, considering a number of 20, 15 and 10 cycles, respectively, we have the normalized coefficients listed in Table 3.

Table 3. Normalized coefficients of interpolation curves for the signal with frequency 5Hz

n	c_5	c_4	c_3	c_2	c_1	c_0
20	1	26.21502	274.2207	1430.919	3724.909	3870.175
15	1	26.23168	274.5002	1432.5	3728.103	3871.018
10	1	26.22083	274.1499	1428.438	3708.43	3837.374

Table 3 shows that the shape of the curve is the same regardless of the number of cycles we consider. The normalized coefficients of interpolation curves for different frequencies, for n=10 cycles, are listed in Table 4.

Table 4. Coefficients for the signal containing n=10 cycles for different frequencies

f [Hz]	C_5	C_4	C_3	C_2	C_1	C_0
5	1	26.22083	274.1499	1428.438	3708.43	3837.374
7	1	36.70863	537.3252	3919.456	14245.53	20637.78
10	1	52.44614	1096.719	11428.57	59340.4	122807.4
14	1	73.41726	2149.301	31355.65	227928.5	660408.8

It is noted that the coefficients are in the ratio:

$$\frac{C_m(f_2)}{C_m(f_1)} = \left(\frac{f_2}{f_1}\right)^{5-m} \tag{9}$$

so that the coefficients for another frequency, but the same number of cycles, can be calculated by the formula:

$$C_m(f_2) = C_m(f_1)\left(\frac{f_2}{f_1}\right)^{5-m} \tag{10}$$

Based on Eq. (8), it is possible to determine the coefficients for any frequency f_2 if those for the frequency f_1 are known. In Table 5 we have the coefficients calculated by this formula for $f = 7\,\text{Hz}$ and in Table 6 these for $f = 10\,\text{Hz}$ if the reference frequency is $f = 5\,\text{Hz}$. These coefficients are compared with the coefficients obtained by interpolating the points which we get from the measurements for the same frequency, both tables indicating also the errors between the sets of coefficients.

Table 5. Coefficients of interpolation curves for the frequencies of 5Hz and 7 Hz

f [Hz]	C_5	C_4	C_3	C_2	C_1	C_0
5	1	26.22083	274.1499	1428.438	3708.43	3837.374
translated from 5 to 7	1	36.70916	537.333899	3919.63460	14246.3030	20638.31
7	1	36.70863	537.3252	3919.456	14245.53	20637.78
Error [%]	0	-0.00146	-0.00161	-0.00455	-0.00543	-0.00263

Table 6. Coefficients of interpolation curves for the frequencies of 5Hz and 10 Hz

f [Hz]	C_5	C_4	C_3	C_2	C_1	C_0
5	1	26.22083	274.1499	1428.438	3708.43	3837.374
translated from 5 to 10	1	52.44166	1096.6	11427.51	59334.87	122796
10	1	52.44614	1096.719	11428.57	59340.4	122807.4
Error [%]	0	0.008531	0.010832	0.009321	0.009321	0.009321

Eq. (9) proves that the shape of the pseudo-sinc function is maintained unaltered, being possible to obtain the coefficients for a given number of cycles and frequency if these are known for a different frequency and number of cycles contained in the signal. This is confirmed by the extremely small errors attained by frequency translation.

5. Conclusions

In this paper we describe the evolution of the coefficient providing the signal amplitude in the case that the analyzed signal length is not a multiple of the period, i.e. it does not comprise an entire number of cycles. We have shown that leakage does not affect the amplitude in concordance with the sinc function, as current literature indicate it, but in concordance to a pseudo-sinc function. Because the spectral lines are not equidistant, the pseudo-sinc function is not symmetric, and actual interpolation methods fail in precise estimating frequencies. However, if the signal contains a large number of cycles, the errors can be diminished.

References

[1] Gillich G.R., Samoilescu G., Berinde F., Chioncel C.P., Experimental determination of the rubber dynamic rigidity and elasticity module by time-frequency measurements, Materiale Plastice **44** (1), 2007, p. 18-21.

[2] Ostachowicz W.M., Krawczuk C., Analysis of the effect of cracks on the natural frequencies of a cantilever beam, Journal of Sound and Vibration, **150** (2), 1991, p. 191–201.

[3] Gillich G.R., Minda P.F., Praisach Z.I., Minda A.A., Natural frequencies of damaged beams - a new approach, Romanian Journal of Acoustics and Vibration, **9** (2), 2012, p. 101-108.

[4] Tufoi M., Hatiegan C., Vasile O., Gillich G.R., Dynamic Analysis of Thin Plates with Defects by Experimental and FEM Methods, Romanian Journal of Acoustics and Vibration **10** (2), 2013, p. 83-88.

[5] Gillich G.R., Praisach Z.I., Detection and Quantitative Assessment of Damages in Beam Structures Using Frequency and Stiffness Changes, Key Engineering Materials, 569, 2013, p. 1013-1020.

[6] Djukanović S., Popović T., Mitrović A., Precise sinusoid frequency estimation based on parabolic interpolation, In Proceedings of the 24th Telecommunications Forum TELFOR, 2016, p. 1-4.

[7] Gillich G.R., Dinamica masinilor. Vibratii, Editura AGIR, Bucuresti, 2005.

[8] Gillich G.R., Mituletu I.C., Signal Post-processing for Accurate Evaluation of the Natural Frequencies. In: R. Yan, X. Chen, S. Mukhopadhyay (eds), Structural Health Monitoring. Smart Sensors, Measurement and Instrumentation, Vol. 26, Springer Cham, 2017, p. 13-37.

[9] Grandke T., Interpolation Algorithms for Discrete Fourier Transforms of Weighted Signals, IEEE Transactions on Instrumentation and Measurement, Vol. 32, 1983, p. 350-355.

[10] Quinn B.G., Estimating Frequency by Interpolation Using Fourier Coefficients, IEEE Transactions on Signal Processing, Vol. 42, 1994, p. 1264-1268.

[11] Jain V.K., Collins W.L., Davis D.C., High-Accuracy Analog Measurements via Interpolated FFT, IEEE Transactions on Instrumentation and Measurement, Vol. 28, 1979, p. 113-122.

[12] Ding K., Zheng C., Yang Z., Frequency Estimation Accuracy Analysis and Improvement of Energy Barycenter Correction Method for Discrete Spectrum, Journal of Mechanical Engineering, **46** (5), 2010, p. 43-48.

[13] Voglewede P., Parabola approximation for peak determination, Global DSP Magazine, **3** (5), 2004, p. 13-17.

[14] Jacobsen E., Kootsookos P., Fast, accurate frequency estimators, IEEE Signal Processing Magazine, **24** (3), 2007, p. 123-125.

[15] Minda A.A., Gillich G.R., Sinc Function based Interpolation Method to Accurate Evaluate the Natural Frequencies, Analele Universitatii Eftimie Murgu Resita, **24** (1), 2017, p. 211-218.

[16] Ntakpe J.L., Gillich N., Gillich G.R., A Practical Method to Increase the Frequency Readability for Vibration Signals, Analele Universitatii Eftimie Murgu Resita, **23** (1), 2016, p. 203-210.

[17] Chioncel C.P, Gillich N., Tirian G.O., Ntakpe J.L., Limits of the discrete Fourier transform in exact identifying of the vibrations frequency, Romanian Journal of Acoustics and Vibration, **12** (1), p. 16–19.

[18] Ntakpe J.L., Gillich G.R., Mituletu I.C., Praisach Z.I., Gillich N., An accurate frequency estimation algorithm with application in modal analysis, Romanian Journal of Acoustics and Vibration, **13** (2), 2016, p. 98-103.

[19] Gillich G.R., Maia N., Mituletu I.C., Praisach Z.I., Tufoi M., Negru I., Early structural damage assessment by using an improved frequency evaluation algorithm, Latin American Journal of Solids and Structures **12** (12), 2015, p. 2311-2329.

[20] Gillich G.R., Mituletu I.C., Negru I., Tufoi M., Iancu V., Muntean F, A Method to Enhance Frequency Readability for Early Damage Detection, Journal of Vibration Engineering& Technologies, **3** (5), 2015, p. 637-652.

[21] Gillich G.R., Mituletu I.C., Praisach Z.I., Negru I., Tufoi M., Method to Enhance the Frequency Readability for Detecting Incipient Structural Damage, Iranian Journal of Science and Technology, Transactions of Mechanical Engineering, **41** (3), 2017, p. 233–242.

[22] Mituletu I.C., Gillich N., Nitescu C.N., Chioncel C.P., A multi-resolution based method to precise identify the natural frequencies of beams with application in damage detection, Journal of Physics: Conference Series, 628(1), 2015, Art. 012020.