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# Exact solution for the natural frequencies of slender beams with an abrupt stiffness decrease

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**Abstract.** In this paper, we introduce a mathematical relation which allows prediction of frequency changes of beams if a damage producing an abrupt stiffness decrease occurs in the structure. This relation is contrived from the energies stored in the beam in the intact, respectively in the damaged state. First, the analysis was performed for the damage reducing the stiffness of the slice subjected to the biggest bending moment. From this analysis, we found the damage severity evolution, reflected in a frequency drop, with the damage depth. Afterward, we replaced this damage on several locations along the beam and quantified the effect upon the natural frequencies for each of them. Based on this analysis, we found out also the effect of damage position upon the natural frequencies. The final relation resulted by aggregating the effect of the severity with that of the position. To prove the reliability of this relation, we tested it against results obtained by means of the finite element method.

**Keywords.** Euler-Bernoulli beam, natural frequency, stiffness, damage assessment.

## 1. Introduction

Damage assessment by vibration-based methods has become an emerging domain in last decades. Natural frequencies are the features most easily extracted from vibration measurements [1]. This qualifies the natural frequency shifts for use in damage assessment methods, even if, in the literature, these are considered as having low sensitivity to damage [2]. Nevertheless, if proper signal processing algorithms are employed, the natural frequencies have less variation due to errors than other modal parameters [3]. It is noticeable that few low frequencies are not

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able to provide information about damage location [4]-[6]. Mass loss due to damage is a factor that introduces complexity in the dynamic behavior of beams, while the loss of mass determines a frequency increase. Another aspect that worth attention if damage assessment is made using frequency shifts is the influence of environmental changes. The mass loss or an unstable environment may provoke frequency alteration, masking damage occurrence or providing false alarms [7].

To overcome these difficulties, we studied in deep the dynamic behavior of beams, focusing on the frequency changes of weak-axis bending vibration modes occurred due to damage initiation and expansion [8]-[10]. In this paper, we present, beside a clear description of the mechanism influencing the frequency shifts due to damage, a simple mathematical relation that allows predicting with high precision frequency changes occurred in beams due to damage.

## 2. Behavior of beam with damage

We conducted the research on prismatic cantilever beams, considering damages reducing the stiffness with various intensities, these being one-by-one positioned in a multitude of locations along the beam. The analysis included ten weak-axis bending vibration modes and was firstly performed by means of the finite element method (FEM). To ensure a fine resolution, we took into consideration 9 levels of damage depth and 200 damage locations. Therefore we attained a good picture about the relation existing between damage parameters and the natural frequency changes. Fig. 1 presents the natural frequencies of the beam in respect to damage location and for three levels of cross-section reduction: 10%, 20% respectively 30%. The frequencies for the intact beam are represented as a reference.

Because we considered numerous damage locations, the points indicating the frequencies are dense and appear as a curve that reflects the frequency drop (or shift). These 2D plots are nominate henceforth frequency shift curves.

From Fig. 1 clearly results that, if the damage affects particular beam slices, no frequency changes are observed irrespective of the damage depth. On the other hand, for certain locations, the frequency drop exhibits local maxima. Both types of points have, for the same mode, similar relative position *x/L* for any slender cantilever beam, irrespective of length and cross-section shape.

These points can be found from the analysis of the stain energy distribution $dU_i$ for various vibration modes, which is expressed

$$dU_i(x) = \frac{1}{2} EI \left[\phi_i''(x)\right]^2 dx \,. \tag{1}$$

In the formula above, $x$ is the distance from the reference point (left fixed beam end), $E$ is the Young's modulus and $I$ is the moment of inertia. The symbol $\phi''(x)$ stays for modal curvature attained as the second derivative of the corresponding mode shape $\phi(x)$ (out of plain displacement at distance $x$) and $i$ for the weak-axis bending vibration mode number.

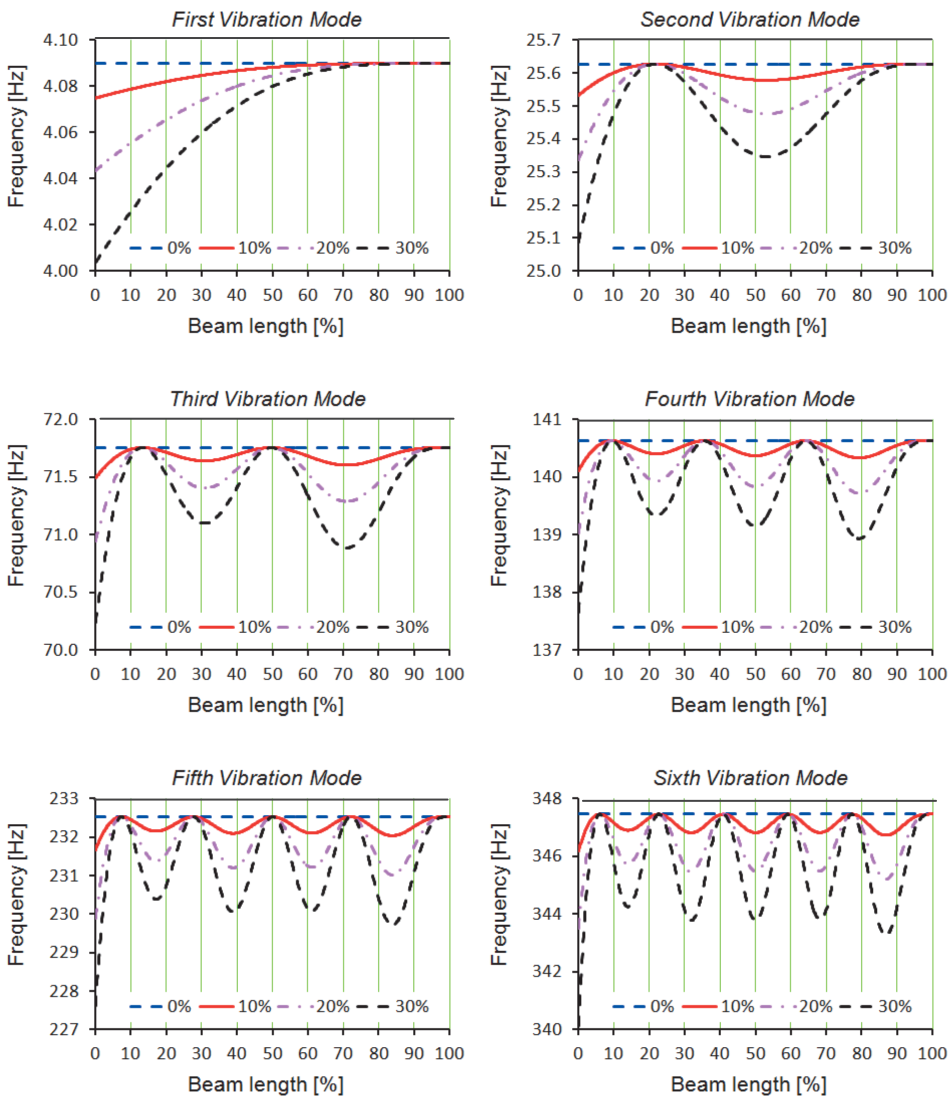


Fig. 1. Natural frequencies versus damage location for three levels of damage depth.

Fig. 2, see also [11], explains how slices located on these particular locations behave. If slices located in the vicinity of inflection points of the mode shapes are affected by damage, as the points marked with a circle in Fig. 2, no frequency drop is remarked. This happens, because in these points the second derivative of the mode shape, i.e. the modal curvature, is null. These slices does not curve, actually being just subject f a rotation as depicted in Fig. 2a, thus no energy is stored here. Since the slice does not contribute to the total potential stored energy, the damage positioned here lead not to frequency changes. In fact, even if a hinge is located here the effect upon the natural frequency will be null. Note that, these points are individualized for each vibration mode, see Table 1.

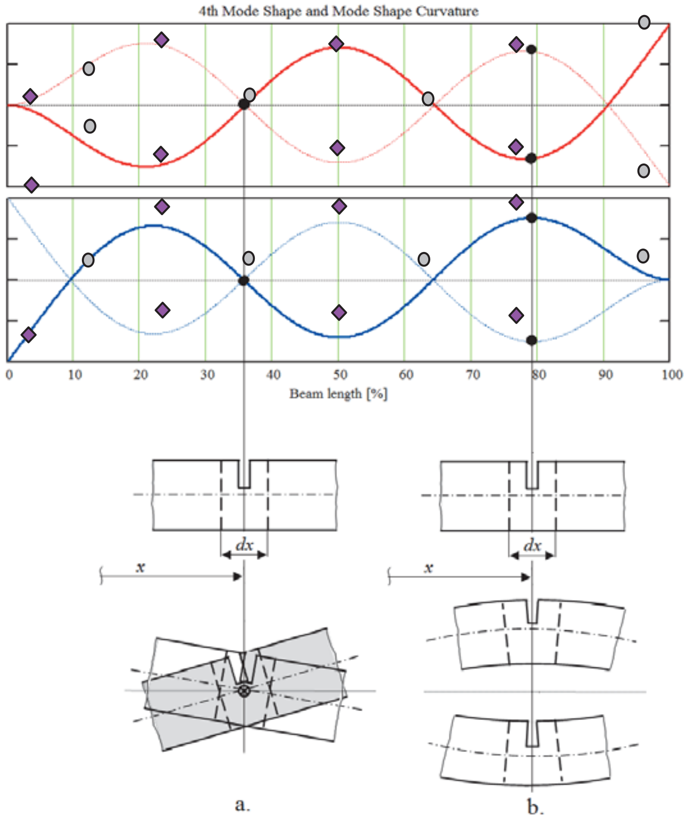


Fig. 2. The behavior of slices in respect to their location on the beam.

Table 1. Inflection point positions *x/L* for the cantilever beam

| | Mode *i* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Position *x/L* | 1 | 0.216 | 0.132 | 0.094 | 0.0735 | 0.060 | 0.050 | 0.044 | 0.038 | 0.034 |
| | | 1 | 0.496 | 0.355 | 0.276 | 0.226 | 0.191 | 0.166 | 0.146 | 0.131 |
| | | | 1 | 0.641 | 0.500 | 0.409 | 0.346 | 0.301 | 0.264 | 0.236 |
| | | | | 1 | 0.721 | 0.590 | 0.499 | 0.435 | 0.382 | 0.342 |
| | | | | | 1 | 0.771 | 0.653 | 0.569 | 0.500 | 0.447 |
| | | | | | | 1 | 0.807 | 0.703 | 0.617 | 0.552 |
| | | | | | | | 1 | 0.836 | 0.735 | 0.657 |
| | | | | | | | | 1 | 0.853 | 0.763 |
| | | | | | | | | | 1 | 0.869 |
| | | | | | | | | | | 1 |

In a different way happens with the slices placed on local maxima of the mode shape, marked with rhombus in Fig. 2. Here the modal curvature achieves also local maxima and the slices are subject of an important bending moment. Due to curving, see Fig. 2b, these slices store an important amount of energy, substantially contributing to the total value of the strain energy. As a consequence, the damage located here produces an important frequency drop.

We observed in Fig. 1 that frequency shifts achieve maxima or do not affect the frequencies exactly in the particular points for each vibration mode. This has leaded us to the conclusion that Eq. (1) can be indeed employed for damage location detection. The consequence is that the mathematical relation expressing the frequency shift in respect to damage parameters should contain the square of modal curvature as a separate term defining the damage location.

Reviewing again Fig. 1 and 2 we can observe that, for all other locations excepting the critical points (nodes or antinodes on the squared modal curvature), the frequency shift takes lower intermediate values, framed by null and those accomplished on locations where bending achieve local maxima.

Table 2. Position *x/L* of the local extrema

| | Mode *i* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Position *x/L* | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | 0.470 | 0.291 | 0.207 | 0.161 | 0.132 | 0.111 | 0.097 | 0.085 | 0.076 |
| | | | 0.692 | 0.500 | 0.389 | 0.318 | 0.269 | 0.234 | 0.205 | 0.184 |
| | | | | 0.779 | 0.611 | 0.499 | 0.423 | 0.368 | 0.323 | 0.289 |
| | | | | | 0.828 | 0.681 | 0.576 | 0.502 | 0.441 | 0.394 |
| | | | | | | 0.859 | 0.730 | 0.636 | 0.558 | 0.500 |
| | | | | | | | 0.881 | 0.770 | 0.676 | 0.605 |
| | | | | | | | | 0.890 | 0.794 | 0.710 |
| | | | | | | | | | 0.910 | 0.816 |
| | | | | | | | | | | 0.919 |

From Fig. 1 can be also observed that the frequency shift curves differ, for the same vibration mode, just by a scaling factor. This, on the other hand, leads to the idea that the mathematical relation expressing the frequency shift in respect to damage parameters should contain the damage severity as a separate term. Intuitively, the frequency drop due to damage with a given severity located anywhere in the beam result as the product between damage location and damage severity.

## 3. The damage severity

Research conducted to find the effect of crack depth on the frequency drop also considers the way how energy is stored in different bending vibration modes. This was made via free end deflection, which is a measure of the energy quantity stored in the beam.

Let us consider that the beam is subjected to dead load, which is a situation similar with the case of forces acting during bending vibration. The maximum deflection in the first mode is similar with that achieved by the mentioned static load. If the beam has a crack of depth $a$ at distance $x$ from the clamped end, it attains at the free end the deflection $\delta_D^x(a)$, while in intact state the deflection is $\delta_U^x(a)$. Obviously, the relation $\delta_U^x(a) < \delta_D^x(a)$ is valid. Imagine an intact beam with same parameters as the original intact beam that achieves same deflection as the cracked beam. This is possible just if it have a lower flexural stiffness, let's say $EI_{eq}$. This imagined intact beam is from now on nominated equivalent beam. For the original cantilever beam subjected just to dead load, the deflection at the free end is:

$$\delta_U^x(a) = \frac{\rho g A L^4}{8EI} = \frac{ML^3}{8EI} \qquad (2)$$

hence, the healthy beam's stiffness is:

$$EI = \frac{ML^3}{8\delta_U^x(a)} \qquad (3)$$

In same conditions, the deflection of the equivalent beam becomes:

$$\delta_D^x(a) = \frac{ML^3}{8EI_{eq}} \qquad (4)$$

and the stiffness of the equivalent beam, behaving as the damaged one, it is:

$$EI_{eq} = \frac{ML^3}{8\delta_D^x(a)} \qquad (5)$$

The well-known relation for the bending modes frequency is

$$f_i = \frac{\lambda_i^2}{2\pi}\sqrt{\frac{EI}{ML^3}} \qquad (6)$$

Therefore, the natural frequencies for the damaged beam, similar with that of the equivalent beam, are:

$$f_{i-D} = \frac{\lambda_i^2}{2\pi}\sqrt{\frac{EI_{eq}}{ML^3}} \qquad (7)$$

From Eq. (6) and (7) results the frequency shift for a beam with abrupt stiffness decrease, i.e. with a breathing crack, as:

$$\Delta f_{i-D} = f_i - f_{i-D} = \frac{\lambda_i^2}{2\pi}\sqrt{\frac{EI}{ML^3}} - \frac{\lambda_i^2}{2\pi}\sqrt{\frac{EI_{eq}}{ML^3}} = \frac{\lambda_i^2}{2\pi\sqrt{ML^3}}\left(\sqrt{EI} - \sqrt{EI_{eq}}\right) \qquad (8)$$

Note that this relation evaluates the frequency drop, but the dimensionless wave numbers $\lambda_i$ are not proper to derive the mode shapes of damaged beam. This means that the equivalent beam reproduces the dynamic behavior of damaged beam in terms of frequency, but can not estimate the beam shape.

In practical applications it is important to know the relative frequency shift, this feature making shifts of more modes better comparable. It is obtained by dividing the frequency shift by the corresponding frequency of the intact beam, resulting:

$$\Delta\bar{f}_{i-D} = \frac{f_i - f_{i-D}}{f_i} = \frac{\dfrac{\lambda_i^2}{2\pi\sqrt{ML^3}}\left(\sqrt{EI} - \sqrt{EI_{eq}}\right)}{\dfrac{\lambda_i^2}{2\pi}\sqrt{\dfrac{EI}{ML^3}}} = \frac{\sqrt{EI} - \sqrt{EI_{eq}}}{\sqrt{EI}} \tag{9}$$

Substituting the flexural stiffness expressions from Eq. (2) and (4), the relative frequency shift in respect to the free end deflections results in:

$$\Delta\bar{f}_{i-D} = \frac{\sqrt{\dfrac{1}{\delta_U^x(a)}} - \sqrt{\dfrac{1}{\delta_D^x(a)}}}{\sqrt{\dfrac{1}{\delta_U^x(a)}}} = \frac{\sqrt{\delta_D^x(a)} - \sqrt{\delta_U^x(a)}}{\sqrt{\delta_D^x(a)}} \tag{10}$$

From Eq. (9) we find the natural frequency of the damaged beam as:

$$f_{i-Di} = f_i - \Delta\bar{f}_{i-D} \cdot f \tag{11}$$

hence, the mathematical relation for the natural frequency of the damaged beam becomes:

$$f_{i-D} = f_i - f_{i-U} \cdot \Delta\bar{f}_{i-D} = f_i\left[1 - \frac{\sqrt{\delta_D^x(a)} - \sqrt{\delta_U^x(a)}}{\sqrt{\delta_D^x(a)}}\right] \tag{12}$$

This mathematical relation is applicable irrespective to the crack location. The particular case in which the crack is located at the fixed end $x = 0$, i.e. the location where the bending moment end consequently the stored energy achieves maxima, is especially interesting. Here, the free end deflection achieves also maxima, as well as the frequency shift. For this particular case, the relative frequency shift is:

$$\Delta\bar{f}_{i-D} = \gamma(a) = \frac{\sqrt{\delta_D^{\max}(a)} - \sqrt{\delta_U^{\max}(a)}}{\sqrt{\delta_D^{\max}(a)}} \tag{13}$$

and reflects the damage severity. The damage severity $\gamma(a)$ attained for the crack positioned at the location where the bending moment achieves maxima is the same for any beam support type. This happens because the numeric value in the deflection expression, individualized by the support types, is simplified in Eq. (13). The results obtained for prismatic beams are comparable with that empirically attained by means of fracture mechanics, presented in the literature [12]-[14]. The clear advantage of the proposed severity estimator is the simplicity.

Knowing that the effect of damage is diminished for other crack locations by the local value of the squared normalized curvature (see section 2), the relative frequency shift for a crack located at distance $x$ from the clamped end becomes:

$$\Delta\bar{f}_{i-D}(x,a)=\frac{\sqrt{\delta_D^x(a)}-\sqrt{\delta_U^x(a)}}{\sqrt{\delta_D^x(a)}}=\frac{\sqrt{\delta_D^{\max}(a)}-\sqrt{\delta_U^{\max}(a)}}{\sqrt{\delta_D^{\max}(a)}}\left[\bar{\phi}_i''(x)\right]^2 \qquad (14)$$

hence

$$\Delta\bar{f}_{i-D}(x,a)=\gamma(a)\cdot\left[\bar{\phi}_i''(x)\right]^2 \qquad (15)$$

As a consequence, the frequency of a beam with a crack of depth $a$, located at distance $x$ from the clamped end becomes:

$$f_{i-D}(x,a)=f_i\left\{1-\frac{\sqrt{\delta_D^{\max}(a)}-\sqrt{\delta_U^{\max}(a)}}{\sqrt{\delta_D^{\max}(a)}}\left[\bar{\phi}_i''(x)\right]^2\right\}=f_i\left\{1-\gamma(a)\cdot\left[\bar{\phi}_i''(x)\right]^2\right\} \qquad (16)$$

This is proved by numerical simulations and experiments for several beam boundary conditions, see [15]-[16].

## 4. Conclusions

This paper presents a precise relation to be used for calculating the natural frequency of beams with breathing cracks. The relation, applicable for beams with any support types, is expressed as $f_{i-D}(x,a)=f_i\left\{1-\gamma(a)\cdot\left[\bar{\phi}_i''(x)\right]^2\right\}$. It permits rapidly creating databases containing any possible damage scenarios, being a great support in damage detection process. A subsequent result is the expression of the severity estimator, known in the literature as the Gillich-Praisach relation, that is $\gamma(a)=\left[\sqrt{\delta_D(a)}-\sqrt{\delta_U(a)}\right]/\sqrt{\delta_D(a)}$. It accurately estimates the maximum effect that a crack can have on the frequency drop due to stiffness loss.

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